f Bd en gesamt.

Sign. mit M Komponenten

↓

M Frequenzen

$$\int_a^b f(t)\,dt$$

Summe der Werte $f(t)$ $\forall t \in [a, b] \subset \mathbb{R}$

(a) DFT eines

1D-Signals $f = f_0, f_1, \ldots$

$\ldots f_{M-1})^T$ besteht:

$$\hat{f}_p := \frac{1}{\sqrt{M}} \sum_{m=0}^{M-1} f_m \exp\left(\frac{-2\pi i\, pm}{M}\right)$$

$p = 0, 1, \ldots, M-1$

(b) Die DFT^{-1}

(diskrete Fourier-Rücktransformation)

$$g_m = \frac{1}{\sqrt{M}} \sum_{p=0}^{M-1} \hat{g}_p \exp\left(\frac{2\pi i p m}{M}\right)$$

$m = 0, 1, \ldots, M-1$

$e^{i\varphi}$

$$x^2 - 4 = 0$$

$x_1 = 2$

$x_2 = -2$

$$x^2 + 4 = 0$$

$x_3 = 2i$ $\qquad i^2 = -1$

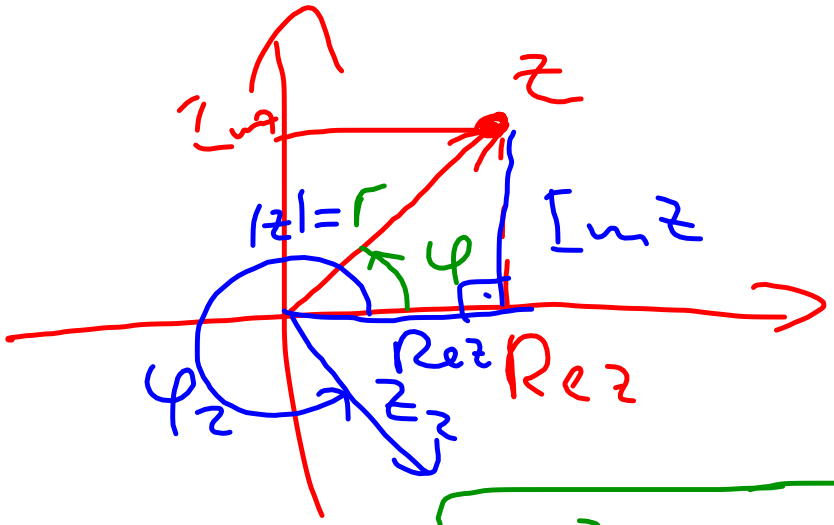
$x_4 = -2i$

$z \in \mathbb{C} \qquad \mathbb{R} \in \mathbb{R} \subset \mathbb{C}$

$z = a + bi = \mathrm{Re}\, z + i\, \mathrm{Im}\, z$

Normalform

$(\mathrm{Re}\, z, \mathrm{Im}\, z) \in \mathbb{R}^2$



$$r = |z| = \sqrt{Re^2 z + Im^2 z}$$

φ $|z|$

$$Re z\,(r, \varphi) = |z| \cos\varphi$$

$$Im z\,(r, \varphi) = |z| \sin\varphi$$

$$z = Re z + i\, Im z =$$

$$= |z| \cos\varphi + i\,|z| \sin\varphi =$$

$$= |z|\,(\cos\varphi + i \sin\varphi)$$

Trigonometr. Form

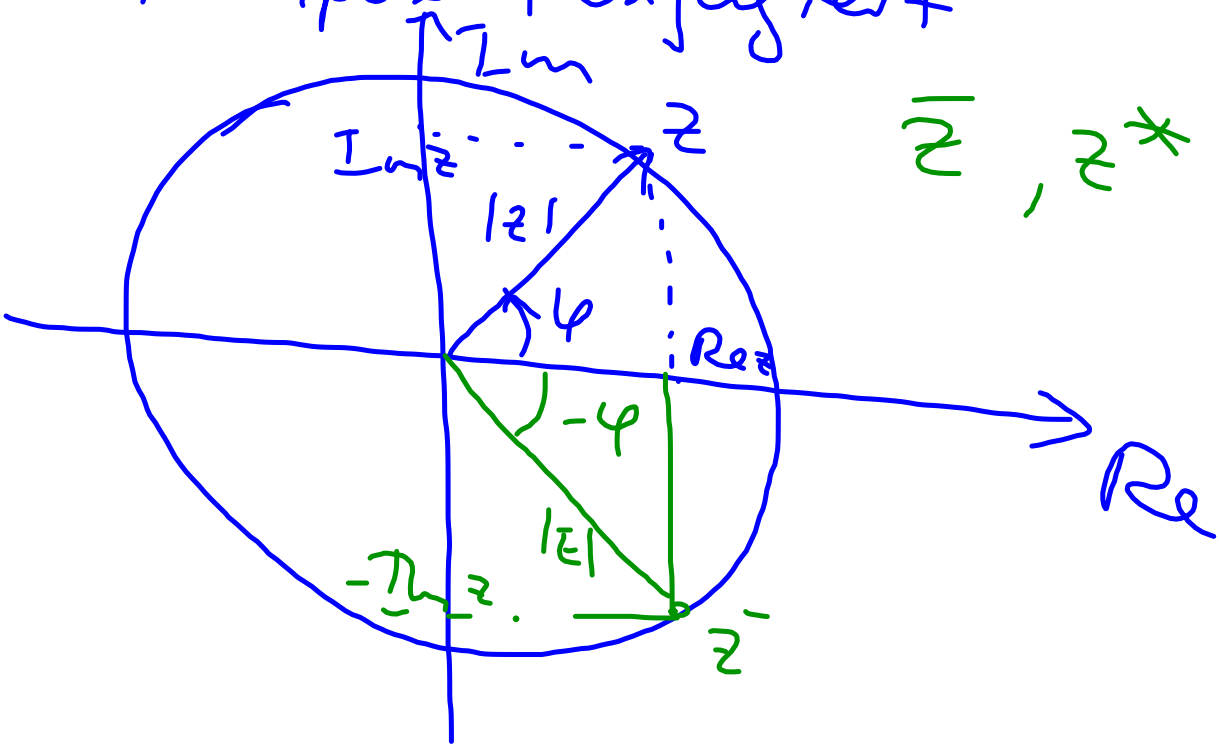
$$z = |z| e^{i\varphi}$$

Euler-Form

$$z = a + ib$$

$$z^* = \bar{z} = a - ib$$

komplex konjugiert



$\bar{z}$, z^*

$$\bar{z} = \mathrm{Re}\, z - i\, \mathrm{Im}\, z$$

$$\mathbb{R} \ni z, \bar{z} \in \mathbb{C}$$

$$\mathbb{R} \ni z \cdot \bar{z} = \mathrm{Re}^2 z + \mathrm{Im}^2 z =$$

$$= |z|^2 = |\bar{z}|^2$$

Try:

$$z = |z|(\cos\varphi + i \sin\varphi)$$

$$\bar{z} = |z|(\cos(2\pi - \varphi) + i \sin(2\pi - \varphi)) =$$

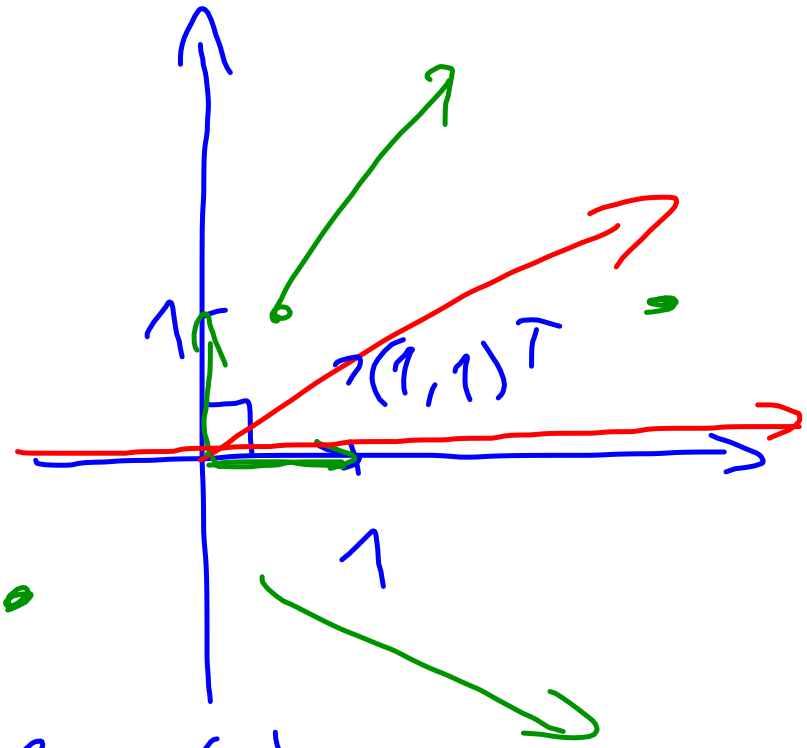
$$= |z|(\cos(-\varphi) + i \sin(-\varphi))$$

$$= |z|(\cos\varphi - i \sin\varphi)$$

$$\bar{z} = |z| e^{i(2\pi - \varphi)} =$$

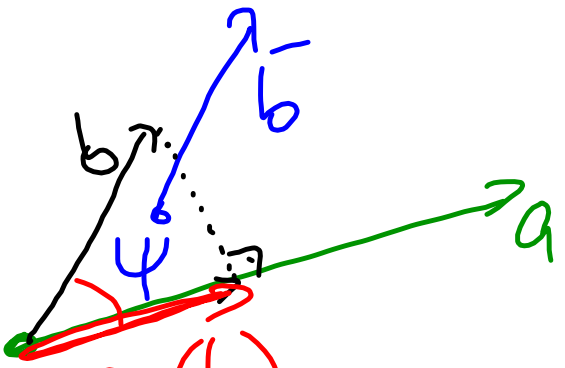
$$= |z| e^{i(-\varphi)} =$$

$$= |z| e^{-i\varphi}$$



$$a = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$b = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$



$\cos\psi = \frac{|P_a(b)|}{|b|}$

$P_{\bar{a}}(\bar{b})$

$\cos\psi = \frac{(a,b)}{|a|\cdot|b|}$

$\bar{b}$

$\frac{|P_a(b)|}{|b|} = \frac{(a,b)}{|a|\cdot|b|} =$

$= \frac{(a,b)}{|a|}$

$\overrightarrow{P_a(b)} = \frac{(\vec{a},\vec{b})}{|\vec{a}|}\,\frac{\vec{a}}{|\vec{a}|}$

$c, d \in \mathbb{R}^n$

$$(c, d) = \sum_{k=1}^{n} c_k d_k$$

Standardskalarprodukt

$f, g \in \mathbb{C}^n$

$$(f, g) = \sum_{k=1}^{n} f_k \overline{g_k}$$

Unitäres Skalarprodukt

(Hermit'sches Sk.-P.)

DFT als Basiswechsel:

$$\langle f, g \rangle := \sum_{m=0}^{M-1} f_m \overline{g_m}$$

$$v_p := \frac{1}{\sqrt{M}} \left(\exp\left(\frac{i 2\pi p 0}{M}\right), \exp\left(\frac{i 2\pi p 1}{M}\right), \exp\left(\frac{i 2\pi p 2}{M}\right), \ldots, \exp\left(\frac{i 2\pi (M-1)}{M}\right) \right)$$

$$p = 0, \ldots, M-1$$

Satz:

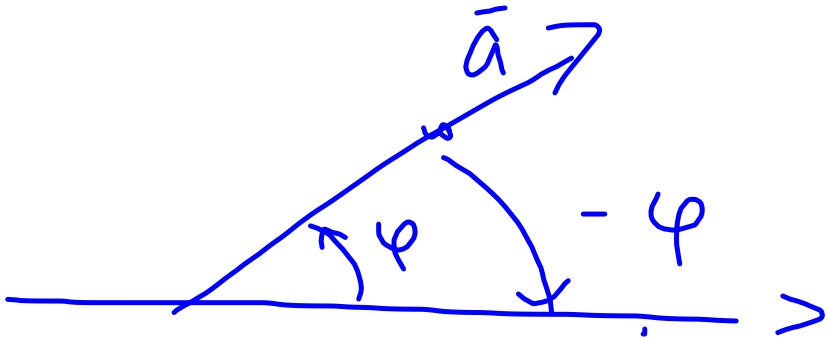
Die M Vektoren

$$v_p := \frac{1}{\sqrt{M}} \left(e^{\frac{i 2\pi p 0}{M}}, e^{\frac{i 2\pi p 1}{M}}, \ldots, e^{\frac{i 2\pi p (M-1)}{M}} \right)^T$$

bilden Bzgl. des unitären Skalarproduktes

$$\langle f | p \rangle := \sum_{k=0}^{M-1} f_k \overline{g_k}$$

eine Orthonormalbasis des M-dimensionalen komplexen Raums.


$\bar{a}$
φ
$-\varphi$

- Stellt man f bzgl. dieser Fourier-Basis dar, so

$$f = \sum_{p=0}^{M} \langle f, v_p \rangle v_p$$

Dies ist gerade die diskrete Fourier-Darstellung m.H. der Fourier-Koeff.

$$\hat{f}_p := \langle f, v_p \rangle$$

Bsp:

$$g = \begin{pmatrix} 2 \\ 3 \\ 4 \\ 4 \end{pmatrix} = (2, 3, 4, 4)^T$$

$$M = 4$$

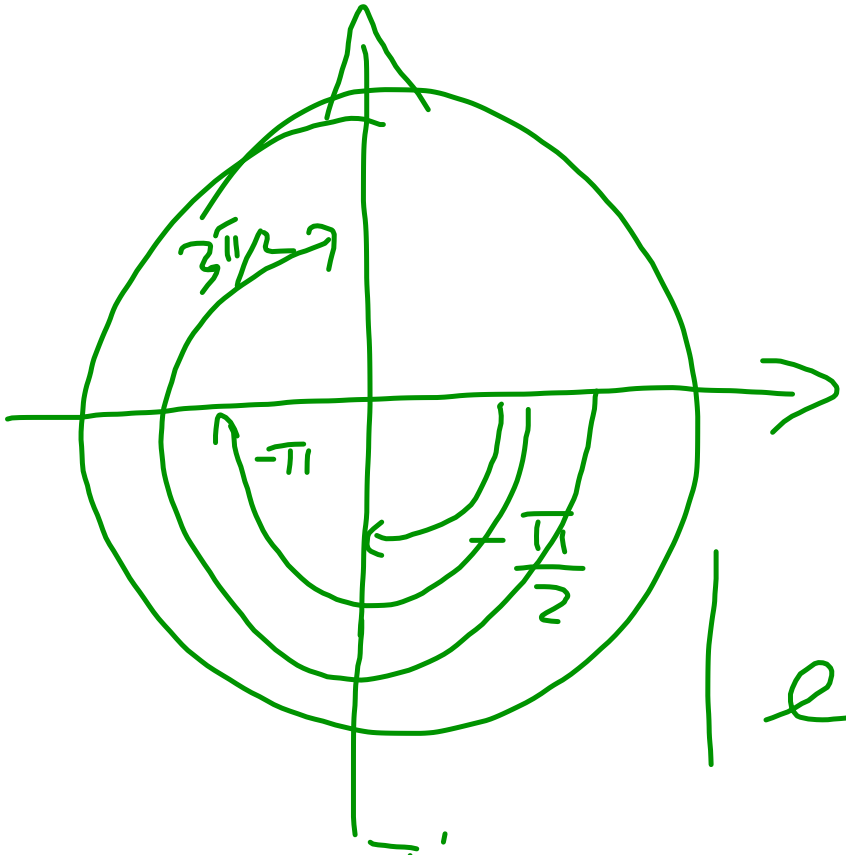
$$\sqrt{M} = 2$$

$$\hat{f}_0 = \frac{1}{2}\sum_{m=0}^{3} f_m \exp\left(\frac{-2\pi i m 0}{4}\right)$$

$$= \frac{1}{2}(2+3+4+4) = \frac{13}{2}$$

$$\hat{f}_1 = \frac{1}{2}\sum_{m=0}^{3} f_m \exp\left(-\frac{2\pi i m 1}{4}\right) =$$

$$= \frac{1}{2}\left(2e^{0} + 3e^{-\frac{\pi i}{2}} + 4e^{-\pi i} + 4e^{-\frac{3\pi i}{2}}\right) = \frac{1}{2}\left(2 + 3(-i) + 4(-1) + 4i\right) = -1 + \frac{1}{2}i = \frac{-2+i}{2}$$



$$|e^{i\varphi}| = 1$$

$$e^{-\frac{\pi i}{2}} = 1\left(\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)\right)$$

$$= -i$$

$$\hat{f}_2 = \frac{1}{2}\sum_{m=0}^{3} f_m e^{-\frac{2\pi i m 2}{4}} =$$

$$= \frac{1}{2}\sum_{m=0}^{3} f_m e^{-\pi i m} =$$

$$= \frac{1}{2}\left(2e^{0} + 3e^{-\pi i} + 4e^{-2\pi i} + 4e^{-3\pi i}\right)$$

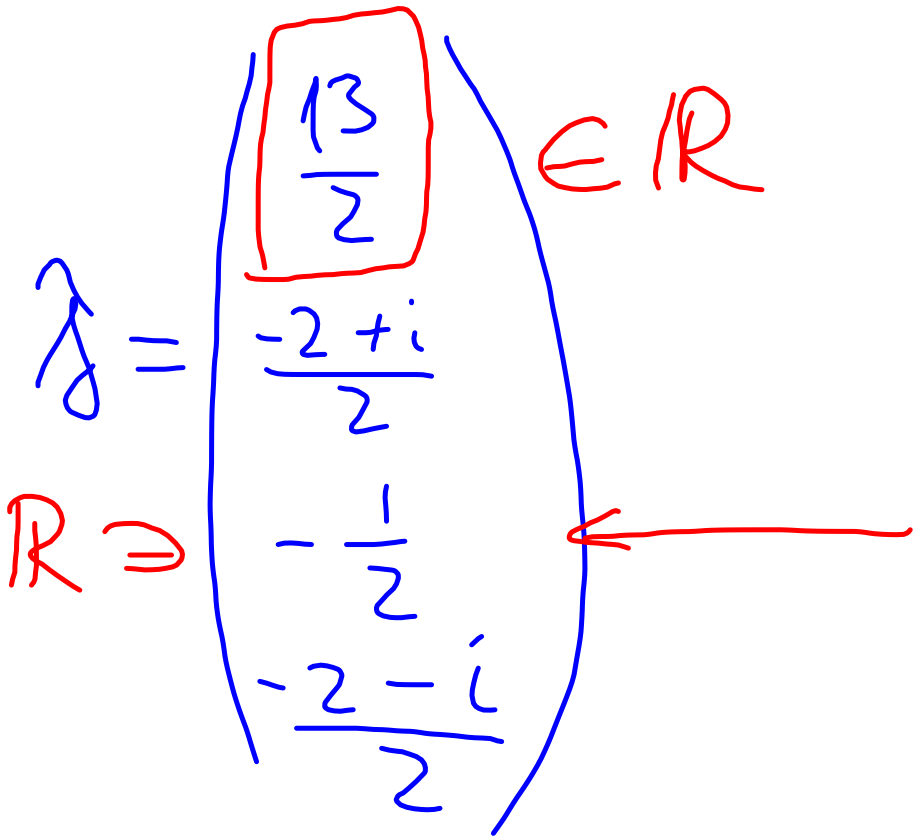
$$= \frac{1}{2}(2 - 3 + 4 - 4) = -\frac{1}{2}$$

$$\hat{f}_3 = \frac{1}{2}\sum_{m=0}^{3} f_m e^{-\frac{2\pi i m 3}{4}} =$$

$$= \frac{1}{2}\left(2e^{0} + 3e^{-\frac{3}{2}\pi i} + 4e^{-3\pi i} + 4e^{-\frac{9}{2}\pi i}\right)$$

$$= \frac{1}{2}\left(2 + 3i - 4 - 4i\right) =$$

$$= \frac{1}{2}\left(-2 - i\right) = \frac{-2-i}{2}$$



$\Rightarrow \hat{f} \in \mathbb{R}^M$

$\Rightarrow$ für $\hat{f}$ brauchen wir (wegen der Sym.) auch M reelle Speicherplätze

Eine diskrete 2D-FT

2D DFT

$$f = (f_{m,n})_{\substack{m=0,\dots,M-1 \\ n=0,\dots,N-1}}$$

$$\hat{f}_{p,q} := \frac{1}{\sqrt{M \cdot N}} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f_{m,n} e^{\left(-\frac{2\pi i m p}{M}\right)} \cdot e^{\left(-\frac{2\pi i n q}{N}\right)},$$

$$q = 0,\dots,N-1$$

$$p = 0,\dots,M-1$$

2D IDFT

2D-DFT^{-1}

$$f_{mn} := \frac{1}{\sqrt{MN}} \sum_{p=0}^{M-1} \sum_{q=0}^{N-1} \hat{f}_{pq} \cdot e^{\left(+\frac{2\pi i \cdot mp}{M}\right)} \cdot e^{\left(+\frac{2\pi i \cdot nq}{N}\right)};$$

$$\hat{f}_{pq} := \frac{1}{\sqrt{MN}} \sum_{m=0}^{M-1} \left(\sum_{n=0}^{N-1} f_{mn} e^{-\frac{2\pi i n q}{N}} \right) \cdot e^{-\frac{2\pi i m p}{M}} =$$

$$= \frac{1}{\sqrt{M}} \sum_{m=0}^{M-1} e^{-\frac{2\pi i m p}{M}} \cdot \left(\frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} f_{mn} e^{-\frac{2\pi i n q}{N}} \right)$$

$$\log_a b = \frac{\log_c b}{\log_c a}$$

Schelle FT: FFT

1D-DFT: (Signal mit Länge M)

$O(M^2) = M^2$ Multipl.

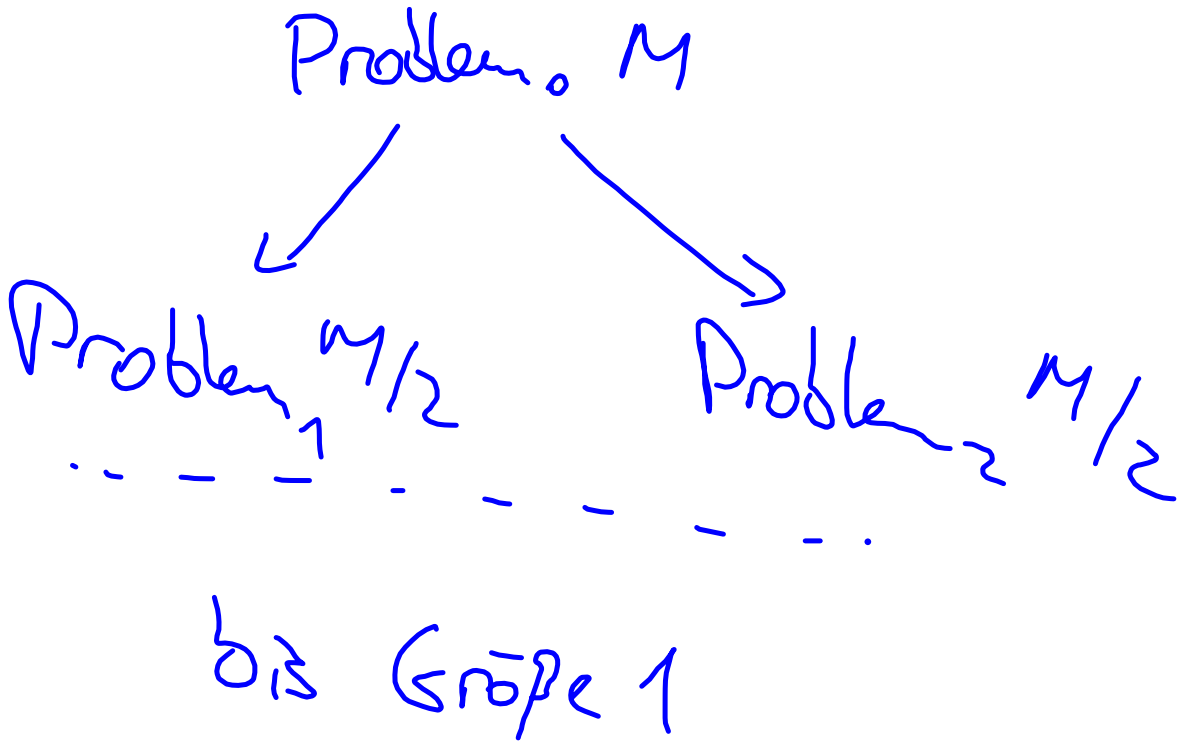
$O(M^2) = M^2 - M$ Additionen

↓

Aufwand

⇒ Gauß ;
Cooley / Tukey 1965

Divide & Conquer:



+ Laufzeit : $O(M \log M)$

+ Black Box - Anwendung
  Zahlreiche Realisierung

– $M = 2^n$

+ www.jjtw.org

Bei Bilder:

- separierbarkeit ausnutzen

→ zus. Speicheraufwand sehr gering

→ sehr gut parallelisierbar auf Shared-Memory-Arch (CUDA)

Die inverse FFT:

1) ersetze die Fourierkoeff. $\hat{f}_p$ durch die komplex konjugierte Werte $\overline{\hat{f}_p} = \hat{f}_p^*$

2) FFT darauf anwenden.

$$\left(\frac{1}{4}, \frac{1}{2}, \frac{1}{4}\right)$$

Gauß-Pyramide

1) $u = (u_0, \ldots, u_{2^N})^T$

2) Signal zum Level k
hat $(2^k + 1)$ Werte, $k \geq 1$
1 Wert, $k = 0$

3) Restriktionsoperator

vom Level k zum Level (k-1)

(Multipl. mit der der $(2^{k-1}+1) \times (2^{k}+1)$ Matrix):

$$R^k := \begin{pmatrix} \frac{2}{3} & \frac{1}{3} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & \cdots & 0 & 0 & 0 & 0 \\ & & & & & & & & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ & & & & & & & & \\ 0 & 0 & 0 & 0 & \cdots & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \frac{1}{3} & \frac{2}{3} \end{pmatrix}$$

$k \geq 2$

$$R^1 := \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$$

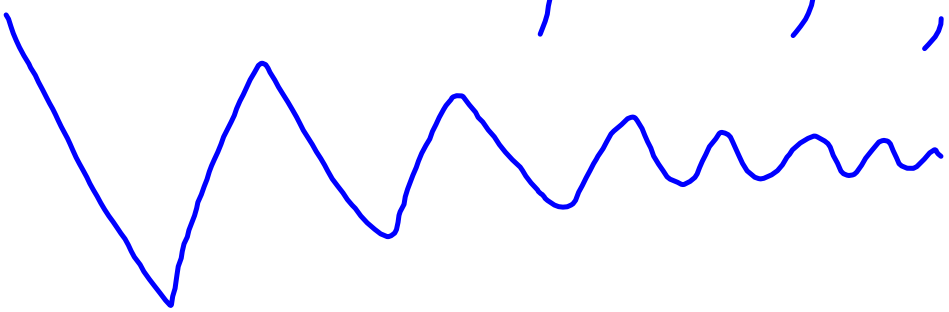
- Die Gaußpyramide

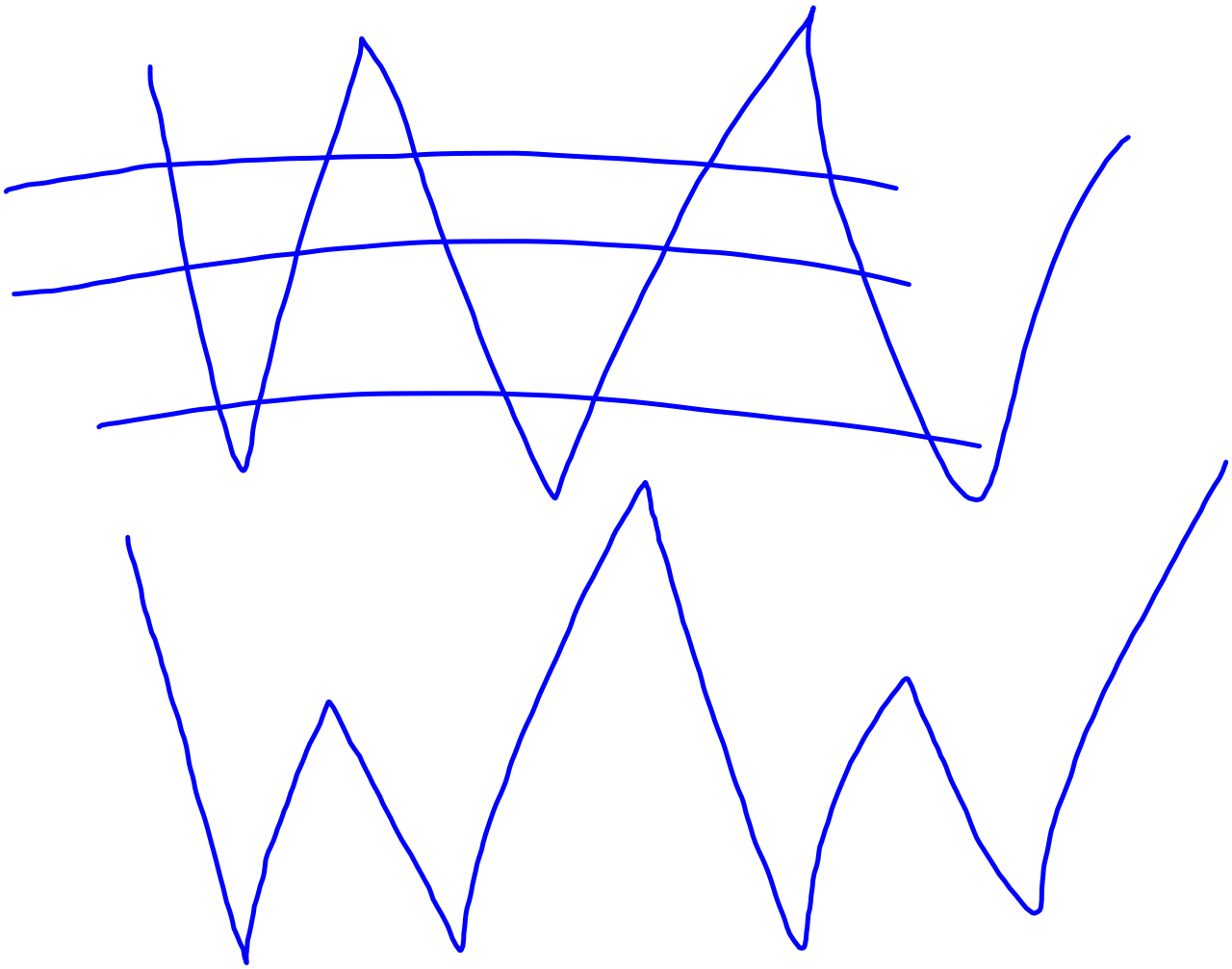
$\{V^N, V^{N-1}, \ldots, V^0\}$ von u

ist definiert durch

$V^N := u;$

$V^{k-1} := R^k V^k; \quad k = N, \ldots, 1$





1D Laplace-Pyramide

- Gesg.: $u = (u_0, \ldots, u_{2^N})^T$
  mit Gaußpyramide $\{u^N, \ldots, u^0\}$
- Interpolationsoperator (Prolongationsoperator) vom Level k zum Level (k+1):

$$P^k := \begin{pmatrix} 1 & 0 & 0 & \cdots & & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & \cdots & & 0 & 0 & 0 \\ 0 & 1 & 0 & \cdots & & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & \cdots & & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \cdots & & 0 & 1 & 0 \\ 0 & 0 & 0 & \cdots & & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & \cdots & & 0 & 0 & 1 \end{pmatrix}$$

$k \geq 1$

$$P^0 := \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$P^k \in \mathbb{R}^{(2^{k+1}+1) \times (2^k+1)}, \quad k \geq 1$$

Die Laplace-Pyramide

$\{w^N, \ldots, w^0\}$ von v

ergibt sich aus

$$w^k := v^k - P^{k-1} v^{k-1},$$

$$w^0 := v^0 \qquad k = N, \ldots, 1$$

---

v^k und $P^{k-1} R^k v^k$

haben den gleichen

mittleren GW

$\Rightarrow$ besondere Behandlung

von den Rändern in R^k