$\cancel{y' = f(x, y)}$

$x_{i+1} = x_i + h$

$f(x_i) = f_i$

h

$f_{i-2} \quad f_{i-1} \quad f_i \quad f_{i+1} \quad f_{i+2}$

$$\underbrace{f_i'' \simeq \alpha f_{i-1} + \beta f_i + \gamma f_{i+1}}$$

$$f_{i-1} = f(x_{i-1}) = f(x_i - h)$$

$$f_{i+1} = f(x_{i+1}) = f(x_i + h)$$

$$f_i = f(x_i) = f(x_i)$$

$$f_{i+1} = f_i + \frac{f_i'}{1!}h + \frac{f_i''}{2!}h^2 + \frac{f_i'''}{3!}h^3 + \frac{f_i''''}{4!}h^4 + O(h^5)$$

$$f_{i-1} = f_i + \frac{f_i'}{1!}(-h) + \frac{f_i''}{2!}(-h)^2 + \frac{f_i'''}{3!}(-h)^3 + \frac{f_i''''}{4!}(-h)^4 + O(h^5)$$

$$f_i'' \approx \alpha f_{i-1} + \beta f_i + \gamma f_{i+1} =$$

$$= \alpha\left(y_i - y_i' h + \frac{y_i''}{2} h^2 - \frac{y_i'''}{6} h^3 + \right.$$

$$\left. + \frac{y_i''''}{24} h^4 + O(h^5)\right) + \beta y_i +$$

$$+ \gamma\left(y_i + y_i' h + \frac{y_i''}{2} h^2 + \frac{y_i'''}{6} h^3 + \right.$$

$$\left. + \frac{y_i''''}{24} h^4 + O(h^5)\right) =$$

$$= y_i (\alpha + \gamma + \beta)$$

$$+ y_i' (-\alpha + \gamma) h +$$

$$+ y_i''(\alpha+\gamma)\frac{h^2}{2} +$$

$$+ y_i'''(-\alpha+\gamma)\frac{h^3}{6} +$$

$$+ y_i^{IV}(\alpha+\gamma)\frac{h^4}{24} + O(h^5)$$

$$\approx 0 \cdot y_i + 0 \cdot y_i' + 1 \cdot y_i''$$

$$\begin{cases} \alpha+\beta+\gamma = 0 \\ -\alpha+\gamma = 0 \\ (\alpha+\gamma)\frac{h^2}{2} = 1 \end{cases}$$

$$\left\{\begin{array}{rcl} \alpha+\beta+\gamma &=& 0 \\ -\alpha \quad\quad +\gamma &=& 0 \\ \alpha \quad\quad +\gamma &=& \frac{2}{h^2} \end{array}\right.$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 2/h^2 \end{array}\right)$$

$$2\gamma = 2/h^2$$

$$\Longrightarrow \gamma = \frac{1}{h^2} = \alpha$$

$$\beta = -2/h^2$$

$$\Longrightarrow y_i'' \approx \frac{y_{i-1} - 2y_i + y_{i+1}}{h^2}$$

$$y_i'' = \frac{1}{h^2}\Bigg(\Big[y_i - y_i' h + \frac{y_i''}{2} h^2 - \frac{y_i'''}{6} h^3 + \frac{y_i^{(4)}}{24} h^4 - \frac{y_i^{(5)}}{120} h^5 + O(h^6) \Big]$$

$$-2 y_i +$$

$$+ \Big[y_i + \frac{y_i'}{1} h + \frac{y_i''}{2} h^2 + \frac{y_i'''}{6} h^3 + \frac{y_i''''}{24} h^4 + \frac{y_i^{(5)}}{120} h^5 + O(h^6) \Big] \Bigg)$$

$$= \frac{1}{h^2}\Big(f_i \cdot 0 + f_i' \cdot 0 + f_i'' h^2 +$$

$$+ f_i''' \cdot 0 + f_i^{(IV)} \cdot \frac{h^4}{12} +$$

$$+ 0 \cdot f_i^{(V)} + O(h^6)\Big) =$$

$$= f_i'' + \underbrace{\frac{f_i^{(IV)}}{12} \cdot h^2}_{O(h^2)} + O(h^4)$$

$$= f_i'' + \underbrace{O(h^2)}_{\text{Fehler}}$$

Ordnung von Fehlertermen
heißt die Konsistenzordnung
der Approximationsformel
(Soll mindestens 1 sein!)

$$f_i'' \approx \frac{f_{i-1} - 2 f_i + f_{i+1}}{h^2}$$

hat die Konsistenzordnung 2.

Standardapproximation

f_i':

$$f_i' = \frac{f_i - f_{i-1}}{h}$$

Appr.-Ord. 1
Rückwärts-
-differenz

$$f_i' = \frac{f_{i+1} - f_i}{h}$$

Appr.-Ordn. 1
Vorwärtsdiffe-
renz

$$f_i' = \frac{f_{i-1} + f_{i+1}}{2h}$$

Zentrale Diff!

Rich-DIF:

$$f_i' \approx \alpha f_{i-1} + \beta f_i$$

$$f_{i-1} = f_i - f_i' h + \frac{f_i''}{2} h^2 - \frac{f_i'''}{6} h^3 + O(h^4)$$

$$f_i' \approx \alpha \left(f_i - f_i' h + O(h^2) \right) + \beta f_i = f_i(\alpha+\beta) + f_i' h(-\alpha) + O(h^2)$$

$$\alpha + \beta$$

$$\alpha + \beta = 0$$

$$-\alpha h = 1$$

$$\begin{cases} \alpha = -1/h \\ \beta = 1/h \end{cases}$$

$$f_i' \simeq \frac{-f_{i-1} + f_i}{h} = \frac{f_i - f_{i-1}}{h}$$

$$f_i' = \frac{1}{h}\left(f_i - \left(f_i - f_i' h + f_i'' \frac{h^2}{2} - \frac{f_i'''}{6} h^3 + O(h^4)\right)\right) =$$

$$= \frac{1}{h}\left[f_i \cdot 0 + f_i' \cdot h - \frac{f_i''}{2} h^2 + O(h^3)\right] =$$

$$= f_i' + \underline{\underline{O(h)}}$$

Zentrale Diff.:

$$f_i' \approx \alpha f_{i-1} + \beta f_{i+1}$$

$$f_{i\pm 1} = f_i \pm \frac{f_i'}{1} h + \frac{f_i''}{2} h^2 \pm \frac{f_i'''}{6} h^3 + \frac{f_i^{(4)}}{24} h^4 \pm \frac{f_i^{(5)}}{120} h^5 + O(h^6);$$

$$y_i' \approx \alpha\left(y_i - y_i' h + \frac{y_i''}{2}h^2 - \frac{y_i'''}{6}h^3 + \right.$$

$$\left. + \ldots \right) +$$

$$+\beta\left(y_i + y_i' h + O(h^2)\right) =$$

$$= y_i(\alpha+\beta) +$$

$$+ y_i'(-\alpha+\beta)h + O(h^2)$$

$$\begin{cases} \alpha+\beta = 0 & \Rightarrow \alpha = -\beta \\ -\alpha+\beta = \frac{1}{h} & \Rightarrow \end{cases} \Downarrow \quad \beta = \frac{1}{2h}$$

$$\Rightarrow f_i' \approx \frac{f_{i+1} - f_{i-1}}{2h};$$

Approx.-Ordnung:

$$f_i' = \frac{1}{2h}\Big[f_i + f_i' h + \frac{f_i''}{2} h^2 + \frac{f_i'''}{6} h^3 + \frac{f_i^{(IV)}}{24} h^4 + O(h^5) -$$
$$- \Big(f_i - f_i' h + \frac{f_i''}{2} h^2 - \frac{f_i'''}{6} h^3 + f_i^{(IV)}/24 \cdot h^4 + O(h^5)\Big)\Big] =$$

$$= \frac{1}{2h}\Big[f_i \cdot 0 + f_i' \cdot 2h + f_i'' \cdot 0 + f_i''' \cdot \frac{h^3}{3} + O(h^4) \Big]$$

$$= f_i' + O(h^2)$$

Approx-Ordn. 2.