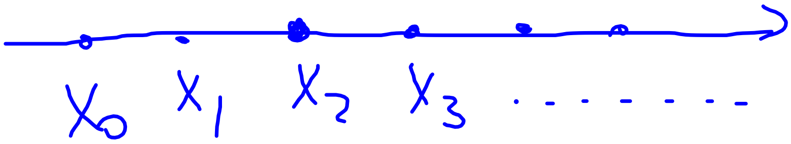
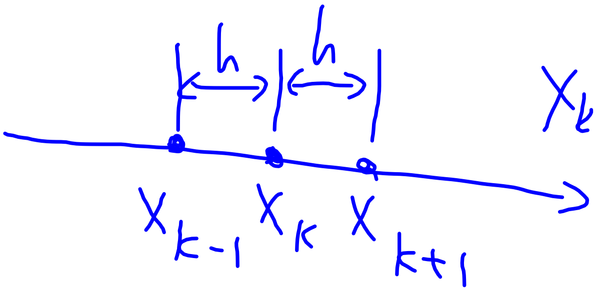
Num. Diff'



$$x_{k+1} = x_k + h$$



$$x_{k-1} = x_k - h$$

$$f'(x) = \lim_{y \to x} \frac{f(x) - f(y)}{x - y} =$$

$$y = x + \Delta x$$

$$y \to x \iff \Delta x \to 0$$

$$= \lim_{\Delta x \to 0} \frac{f(x) - f(x \pm \Delta x)}{\Delta x}$$

$$\text{„}\Delta x \rightsquigarrow h \cdot m, \quad m \in \mathbb{N}$$

$$\Delta x \to 0\text{"} \rightsquigarrow m \searrow 1$$

$$f'(x_k) \approx \frac{f(x) - f(x \pm \Delta x)}{\Delta x}$$

$$\Rightarrow \quad D f_k \approx f'(x_k) \equiv f'_k \approx$$

$$f_k = f(x_k) \approx \frac{f(x_k + h) - f(x_k)}{h} \approx$$

$$\approx \frac{f(x_k) - f(x_k - h)}{h} \approx$$

$$\approx \frac{f(x_k + h) - f(x_k - h)}{2h}$$

$$D^+ f_k = \frac{f_{k+1} - f_k}{h}$$

Vorwärtsdiff.

Rückw-Diff:

$$D^- f_k = \frac{f_k - f_{k-1}}{h}$$

Zentr. Diff:

$$D^0 f_k = \frac{f_{k+1} - f_{k-1}}{2h}$$

2. Abl. : $f'' = (f')'$

$$f''_k \approx D^+(D^- f_k) =$$

$$= \frac{f_{k+1} - 2f_k + f_{k-1}}{h^2}$$

Bsp-Alg:

Man approximiere y''_k mit Hilfe von y_{k-1}, y_k, y_{k+1}

Linearkombination:

α, β, γ werden gesucht, so dass

$$\boxed{\underline{\underline{y''_k}} \approx \alpha y_{k-1} + \beta y_k + \gamma y_{k+1}}$$

$$y_{k-1} = y(x_{k-1}) = y(x_k - h)$$
$$y_k = y(x_k + 0)$$
$$y_{k+1} = y(x_k + h)$$

$$g(x = x_0 + \Delta x) =$$
$$= \sum_{m=0}^{\infty} \frac{y^{(m)}(x_0)}{m!} \underbrace{(x - x_0)^m}_{(\Delta x)^m} =$$
$$= y(x_0) + y'(x_0)\Delta x +$$
$$+ y''(x_0)\frac{(\Delta x)^2}{2} + \frac{y'''(x_0)}{6}(\Delta x)^3 +$$
$$+ \frac{y^{(iv)}(x_0)}{24}(\Delta x)^4 + \frac{y^{(v)}(x_0)}{120}(\Delta x)^5 +$$
$$+ O\left(\Delta x^{(n+1)}\right)$$

$$y_{k-1} = y_k - y'_k \cdot h + \frac{y''_k}{2} \cdot h^2 - \frac{y'''_k}{6} h^3 + \frac{y^{IV}_k}{24} h^4 - \frac{y^{(5)}_k}{120} h^5 + O(h^6);$$

$$y_k = y_k;$$

$$y_{k+1} = y_k + y'_k \cdot h + y''_k/2 \cdot h^2 + \frac{y'''_k}{6} h^3 + \frac{y^{IV}_k}{24} \cdot h^4 + \frac{y^{V}_k}{120} \cdot h^5 + O(h^{6}$$

$$y_k'' \simeq \alpha y_{k-1} + \beta y_k + \gamma y_{k+1} =$$

$$= \alpha \Big(y_k - y_k' h + y_k''/2 \cdot h^2 - \frac{y_k'''}{6} h^3 +$$

$$+ \frac{y_k^{IV}}{24} h^4 - \frac{y_k^{V}}{120} h^5 + O(h^6) \Big) +$$

$$+ \beta y_k +$$

$$+ \gamma \Big(y_k + y_k' h + \frac{y_k''}{2} h^2 + \frac{y_k'''}{6} h^3 +$$

$$+ \frac{y_k^{(IV)}}{24} h^4 + \frac{y_k^{V}}{120} h^5 + O(h^6) \Big) =$$

$$= f_k(\alpha+\beta+\gamma) + f_k'(-\alpha h+\gamma h) +$$

$$+ f_k''\left(\alpha \frac{h^2}{2} + \gamma \frac{h^2}{2}\right) +$$

$$+ f_k'''\left(-\alpha \frac{h^3}{6} + \gamma \frac{h^3}{6}\right) +$$

$$+ f_k^{IV}\left(\alpha \frac{h^4}{24} + \gamma \frac{h^4}{24}\right) + \ldots$$

$$= f_k(\alpha+\beta+\gamma) + f_k'(\gamma-\alpha)h +$$

$$+ f_k''(\alpha+\gamma)\frac{h^2}{2} + f_k'''(\gamma-\alpha)\frac{h^3}{6} +$$

$$+ f_k^{IV}(\alpha+\gamma)\frac{h^4}{24} + f_k^{V}(\gamma-\alpha)\frac{h^5}{120} +$$

$$+ O(h^6) = 1 \cdot f_k'' + 0 \cdot f_k + 0 \cdot f_k' +$$

$$+ \ldots 0 \cdot f_k^{m} + \ldots$$

Koeff-Vergleich

$h \neq 0$

$$\alpha + \beta + \gamma = 0$$

$$(\gamma - \alpha)\cancel{h} = 0$$

$$(\alpha + \gamma)\frac{h^2}{2} = 1$$

$$\begin{cases} (\gamma - \alpha)\dfrac{h^{2n+1}}{(2n+1)!} = 0 \\ (\gamma + \alpha)\dfrac{h^{2n}}{(2n)!} = 0 \end{cases}$$

$\Rightarrow$ <u>LGS</u>:

$$\begin{cases} \alpha+\beta+\gamma = 0 \\ -\alpha + \gamma = 0 \\ \alpha + \gamma = \frac{2}{h^2} \end{cases} \quad \left| \; \begin{array}{l} \alpha = \gamma \\ \text{"+"} \Downarrow \\ 2\gamma = \frac{2}{h^2} \end{array} \right.$$

$$\Downarrow$$

$$\gamma = \frac{1}{h^2} = \alpha$$

$$\Rightarrow \beta = -\gamma - \alpha = -\frac{2}{h^2}$$

$$f''_k \approx \alpha f_{k-1} + \beta f_k + \gamma f_{k+1} =$$

$$= \frac{1}{h^2} f_{k-1} - \frac{2}{h^2} f_k + \frac{1}{h^2} f_{k+1} =$$

$$= \frac{f_{k-1} - 2f_k + f_{k+1}}{h^2};$$

Genauigkeit

$$f_k'' = \frac{f_{k-1} - 2f_k + f_{k+1}}{h^2} =$$

$$= \frac{1}{h^2}\Big[\cancel{f_k} - \cancel{f_k' h} + f_k'' h^2/2 - \cancel{\frac{f_k'''}{6}} h^3 +$$

$$+ \frac{f_k^{IV}}{24} h^4 - \frac{f_k^{V}}{120} h^5 + O(h^6) -$$

$$- \cancel{2 f_k} +$$

$$+ \cancel{f_k} + \cancel{f_k' h} + f_k'' \frac{h^2}{2} + \cancel{\frac{f_k'''}{6}} h^3 + \frac{f_k^{(4)}}{24} h^4 +$$

$$+ \frac{f_k^{(5)}}{120} h^5 + O(h^6)\Big] =$$

$$= \frac{1}{h^2}\Big[f_k''\Big(\frac{h^2}{2} + \frac{h^2}{2}\Big) + \frac{f_k^{(IV)}}{24}\big(h^4 + h^4\big) +$$

$$+ O(h^6)\Big] =$$

$$= \frac{1}{h^2}\left[y_k'' \cdot h^2 + O(h^4) \right] =$$

$$= y_k'' + \underbrace{\frac{O(h^4)}{h^2}}_{= O(h^2)} =$$

$$= y_k'' + O(h^2) \approx y_k''$$

Fehler: $\underline{\underline{O(h^2)}}$

Bsp (Probeklausur)

Gegeben: $x_{k+1} - x_k = h$

f_{k-3}, f_k, f_{k+3}

Gesucht: f''_k

$$f''_k = \alpha f_{k-3} + \beta f_k + \gamma f_{k+3}$$

Taylor:

$$y_{k\pm 3} = y_k \pm y_k' (3h) + \frac{y_k''}{2}(3h)^2 \pm \frac{y_k'''}{6}(3h)^3 + \frac{y_k^{IV}}{24}(3h)^4 \pm \frac{y_k^{(V)}}{120}(3h)^5 + O(h^6);$$

$$y_k'' = y_k(\alpha+\beta+\gamma) + y_k'(-\alpha\cdot 3h + \gamma\cdot 3h) + y_k''\left(\alpha\frac{9}{2}h^2 + \gamma\frac{9}{2}h^2\right) + y_k'''\left(-\alpha\frac{9}{2}h^3 + \gamma\frac{9}{2}h^3\right)\ldots\ldots$$

Koeff.-Vergleich:

$$\begin{cases} \alpha+\beta+\gamma=0 \\ (\gamma-\alpha)\,3h=0 \\ (\alpha+\gamma)\,\frac{9}{2}h^2=1 \end{cases}$$

$$\alpha+\beta+\gamma=0$$
$$-\alpha+\gamma=0$$
$$\alpha+\gamma=\frac{2}{9h^2}$$

$$\Rightarrow \gamma=\alpha=\frac{1}{9h^2}$$

$$\beta=-\gamma-\alpha=-\frac{2}{9h^2}$$

$$y''_k \approx \frac{y_{k-3}-2y_k+y_{k+3}}{9h^2}$$

$$y_k'' = \frac{1}{9h^2}\Big[\cancel{y_k} - \cancel{y_k'(3h)} + y_k''\frac{(3h)^2}{2} -$$

$$- \cancel{y_k'''\frac{27}{6}h^3} + y_k^{(IV)}\frac{81}{24}h^4 -$$

$$- \cancel{y_k^{(V)}\frac{243}{120}}h^5 - \cancel{2y_k} +$$

$$+ \cancel{y_k} + \cancel{y_k'(3h)} + y_k''\frac{9h^2}{2} +$$

$$+ \cancel{y_k'''\frac{27h^3}{6}} + y_k^{(IV)}\frac{81}{24}h^4 + \cancel{y_k^{(V)}\frac{243}{120}}$$

$$+ O(h^6)\Big] =$$

$$= \frac{1}{9h^2}\left[0 \cdot y_k + y_k'' \cdot 9h^2 + y_k^{IV} \frac{162}{24} h^4 + O(h^6) \right] = y_k'' + \frac{O(h^4)}{9h^2} =$$

$$= y_k'' + O(h^2)$$

Fehler

$\Longrightarrow$ Konsistenzordnung ist $\underline{\underline{2}}$

$$y'_k = \alpha y_{k-2} + \beta y_{k-1} +$$
$$+ \gamma y_k + \delta y_{k+1} + \varepsilon y_{k+2}$$