Best. Int

unb. Int.

Rechn. Regel.

Part. Integr.

1.8. Integration durch Substitution

Bsp. $\int \underline{\underline{\cos}} \, (3x+7) \, dx$

Substitution:

$u = 3x + 7$

$\int \cos \underline{\underline{u}} \, \underline{\underline{dx}}$

$u = 3x + 7$

$u' = (3x+7)'$

$\frac{du}{dx} = 3$

$du = 3dx$

$dx = \frac{1}{3} du$

$x = \frac{1}{3}(u - 7)$

$1 = \frac{1}{3}(u' - 7)' =$

$= \frac{1}{3} u'$

$dx = \frac{1}{3} du$

$$\left| \begin{array}{l} u = 3x + 7 \\ dx = \frac{1}{3}\, du \end{array} \right|$$

$$\int \cos(3x+7)\,dx =$$

$$= \int \cos u \cdot \frac{1}{3} \cdot du =$$

$$= \frac{1}{3} \int \cos u \, du = \frac{1}{3} \sin u + C =$$

Rücksubst.:

$$= \frac{1}{3} \sin(3x+7) + C$$

Bsp:

$$\int_0^1 \sqrt{3x+1}\,dx$$

$$\int_{x=0}^{x=1} \sqrt{3x+1}\,dx$$

Best. Integral:

1) a) Stammfunkt. (unb. Int.)

b) Auswertung

2) Mitsubst. der Grenzen

1) oder 2)

1) $\int_0^1 \sqrt{3x+1}\,dx$

a) $\int \sqrt{3x+1}\,dx =$ $\left| \begin{array}{l} u = 3x+1 \\ u' = \frac{du}{dx} = (3x+1)' \\ dx = \frac{1}{3}\,du \end{array} \right|$

$= \int \sqrt{u}\cdot\frac{1}{3}\cdot du = \frac{1}{3}\int u^{\frac{1}{2}}du =$

$= \frac{1}{3}\cdot\frac{2}{3}\,u^{3/2} + C = \frac{2}{9}u^{3/2} + C =$

$= \frac{2}{9}(3x+1)^{3/2} + C$;

b) $\int_0^1 \sqrt{3x+1}\,dx = F(x)\Big|_0^1 =$

$= \frac{2}{9}(3x+1)^{3/2}\Big|_0^1 = \frac{2}{9}(3\cdot 1+1)^{3/2} -$

$- \frac{2}{9}(3\cdot 0+1)^{3/2} = \frac{2}{9}\left(4^{3/2} - 1\right) = \frac{14}{9}$

2) $\int_0^1 \sqrt{3x+1}\, dx = \left| \begin{array}{l} u = 3x+1 \\ dx = \frac{1}{3} du \\ u(1) = 3\cdot 1+1 = \\ \quad = 4 \\ u(0) = 3\cdot 0+1 \\ \quad = 1 \end{array} \right| =$

$$= \frac{1}{3}\int_1^4 \sqrt{u}\, du = \frac{1}{3}\cdot\frac{2}{3} u^{3/2}\Big|_1^4 =$$

$$= \frac{2}{9} u^{3/2}\Big|_1^4 = \frac{2}{9}\left(4^{3/2} - 1^{3/2}\right) =$$

$$= \frac{14}{9}$$

Allgemein:

Substitution Typ 1

Gegeben sei $\int f(ax+b)\,dx$, wobei die Stammfunktion von f bekannt (oder leicht zu berechnen) ist.

Methode:

unbestimmtes Integral:

$$\int f(ax+b)\,dx = \frac{1}{a}\int f(u)\,du =$$

$$= \left| u = ax+b, \quad u' = a = \frac{du}{dx} \Rightarrow dx = \frac{1}{a}\,du \right| =$$

$$= \frac{1}{a}F(u) + c \overset{\text{Rücksubst.}}{=} \frac{1}{a}F(ax+b) + c,$$

$c \in \mathbb{R}$

best. Integral:

(i) 1) unbest. Integral berechnen
2) Grenzen einsetzen

oder

(ii) $\int_{x_1}^{x_2} f(ax+b)\,dx \ominus$

Subst. $u = ax+b$
$dx = \frac{1}{a}\,du$

$u = ax+b$: $x_1 \longmapsto ax_1 + b = u_1$
$x_2 \longmapsto ax_2 + b = u_2$

$$\ominus \frac{1}{a} \int_{u_1 = ax_1+b}^{u_2 = ax_2+b} f(u)\,du = \frac{1}{a} F(u) \Big|_{u_1}^{u_2} =$$

$$= F(ax_2+b) \cdot \frac{1}{a} - \frac{1}{a} F(ax_1+b).$$

Substitution Typ 2

Geg.:

$$\int f'(x) \cdot f(x)\, dx$$

unbest.:

$u = f(x)$

$u' = f'(x)$

$du = f'(x)\, dx$

$$\int \underbrace{f(x)}_{u} \cdot \underbrace{\left(f'(x)\, dx\right)}_{du} =$$

$$= \int u\, du = \frac{1}{2} u^2 + C$$

$$= \frac{1}{2}\left(f(x)\right)^2 + C$$

best. Integral

$$\int_a^b f'(x) \cdot f(x)\, dx =$$

$$= \int_{f(a)}^{f(b)} u\, du =$$

$$= \frac{1}{2} u^2 \Big|_{f(a)}^{f(b)} =$$

$$= \frac{1}{2} \left(f(x) \right)^2 \Big|_a^b =$$

$$= \frac{1}{2} \left(f^2(b) - f^2(a) \right).$$

Bsp:

$$\int \frac{\ln x}{x}\,dx = \int \ln x \cdot \underbrace{\left(\frac{1}{x}\,dx\right)}_{=\ln'(x)\cdot dx} =$$

$$= \left|\begin{array}{l} u = \ln x \\ du = \frac{1}{x}\,dx \end{array}\right| =$$

$$= \frac{1}{2}\ln^2 x + C$$

Aufgabe! $\int \sin x \cos x \; dx$

1) partiell
2) Substit. Typ 2
3) Trig. Formel $\sin 2\alpha$ und Substit. Typ 1

Substitution Typ 3

$$\int \frac{f'(x)}{f(x)}\, dx$$

Subst. : $u = f(x)$

<u>unbestimmt!</u>

$du = f'(x)\, dx$

$$\int \frac{f'(x)}{f(x)}\, dx = \int \frac{f'(x)\, dx}{f(x)} =$$

$$= \int \frac{du}{u} = \ln|u| + c =$$

$$= \ln|f(x)| + c, \; c \in \mathbb{R}$$

bestimmt! analog zum

Typ 2

Bsp! $\displaystyle\int_0^1 \frac{4x+4}{x^2+2x+3}\,dx$

$$(x^2+2x+3)' = 2x+2.$$

$$\Rightarrow \int_0^1 \frac{(4x+4)\,dx}{x^2+2x+3} = \int_0^1 \frac{2\left[(2x+2)\,dx\right]}{x^2+2x+3} =$$

$u = x^2+2x+3$ | $u(1) = 1^2+2\cdot 1+3 = 6$

$du = (2x+2)\,dx$ | $u(0) = 3$

$$= 2\int_{\cancel{0}}^{\cancel{1}} \frac{du}{u} = 2\int_3^6 \frac{du}{u} =$$

$$= 2\ln|u|\Big|_3^6 = 2(\ln 6 - \ln 3) =$$

$$= \underline{2\ln 2}.$$

Substitution Typ 4

$$\int f'(x) \cdot g(f(x))\,dx$$

Stammfunktion $G(x)$ von $g(x)$ sei bekannt

unbestimmt:

$$\int f'(x)\, g(f(x))\,dx$$

$$f(x) = u$$

$$u' = f'(x) = \frac{du}{dx}$$

$$du = f'(x) \cdot dx$$

$$\int f'(x) \cdot g(f(x))\,dx =$$

$$= \int g(f(x)) \cdot (f'(x)\,dx) =$$

$$= \int g(u) \cdot du = G(u) + c =$$

$$= G(f(x)) + c,\ c \in \mathbb{R}$$

best.: analog

Bsp!

$$\int \cos x \cdot e^{\sin x}\, dx =$$

$\sin'(x)$ $\quad$ $g(\sin x)$

$$= \int f' \cdot g(f)\, dx =$$

$u = \sin x$
$du = \cos x\, dx$

$$= \int e^{u}\, du = e^{u} + C =$$

$$= e^{\sin x} + C, \quad C \in \mathbb{R}$$

Bsp!

$$\int_0^1 2x\sqrt{x^2+7}\,dx$$

Subst!

$$u = x^2+7$$

$$du = 2x\,dx$$

$$u(0) = 7$$

$$u(1) = 1^2+7 = 8$$

$$\int_0^1 2x\sqrt{x^2+7}\,dx =$$

$$= \int_7^8 \sqrt{u}\,du =$$

$$= \int_7^8 u^{\frac{1}{2}}\,du = \frac{2}{3}u^{\frac{3}{2}}\Big|_7^8 =$$

$$= \frac{2}{3}\left(8^{3/2} - 7^{3/2}\right)$$

$$\int_0^1 \frac{1}{(2x+7)^4}\,dx \qquad \text{Typ 1}$$

$$\int \sqrt{x-2}\,dx \qquad \text{Typ 1}$$

$$\int x e^{-x}\,dx \qquad \text{part. Int.}$$

Weitere Subst.-Typen

in Prüfung