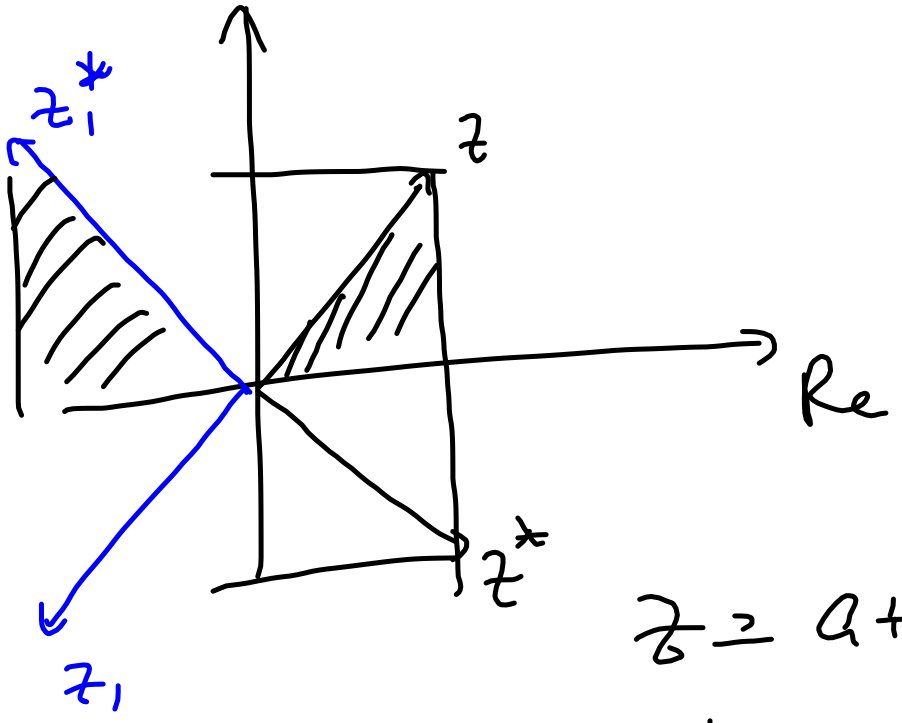


$z = a + ib$

$z^* = a - ib$

$$(z^*)^* = z$$

$r, \varphi \qquad |z|, \varphi$

$$z = |z|(\cos\varphi + i \sin\varphi)$$

$$\operatorname{Re} z = |z| \cos\varphi$$

$$\operatorname{Im} z = |z| \sin\varphi$$

$$|z| = \sqrt{\operatorname{Re}^2 z + \operatorname{Im}^2 z}$$

$$\varphi = \arccos \frac{\operatorname{Re} z}{|z|} = \arcsin \frac{\operatorname{Im} z}{|z|} =$$

$$= \operatorname{arctan} \frac{\operatorname{Im} z}{\operatorname{Re} z}$$

bis auf Quadrant

1.2.4. Die Eulerform einer komplexen Zahl

Satz: Es gilt:

$$e^{i\varphi} = \cos\varphi + i\sin\varphi$$

Folgerung:

$$\cos\varphi = \frac{e^{i\varphi} + e^{-i\varphi}}{2}$$

$$\sin\varphi = \frac{e^{i\varphi} - e^{-i\varphi}}{2i}$$

Beweis:

$e^{i\varphi}$; $\cos\varphi$; $\sin\varphi$

werden als Taylorreihen

um $x_0 = 0$:

TR für $f(x)$ um x_0:

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k$$

für $x_0 = 0$:

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k$$

TR f-r e^x:

$f'(x) = e^x$

$f''(x) = (f'(x))' = (e^x)' = e^x$

- - - - - -

$\forall k: \quad f^{(k)}(x) = e^x$

$f^{(k)}(0) = e^0 = 1$

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} =$$

$$= 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \dots\dots$$

Analog:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

mit $x = i\varphi$:

$$e^{i\varphi} = 1 + i\varphi + \frac{(i\varphi)^2}{2} + \frac{(i\varphi)^3}{3!} + \frac{(i\varphi)^4}{4!} + \frac{(i\varphi)^5}{5!} + \dots =$$

$$= 1 + i\varphi + \frac{i^2}{2}\varphi^2 + \frac{i^3}{3!}\varphi^3 +$$

$$+ \frac{i^4}{4!}\varphi^4 + \frac{i^5\varphi^5}{5!} + \ldots\ldots = e$$

$$i^1 = i$$

$$i^2 = -1$$

$$i^3 = i^2 \cdot i = -i$$

$$i^4 = i^2 \cdot i^2 = 1$$

$$i^5 = i^4 \cdot i = i$$

$$i^6 = i^4 \cdot i^2 = -1$$

$$\downarrow$$

$$1$$

$$= 1 + i\varphi - \frac{1}{2}\varphi^2 - i\frac{1}{3!}\varphi^3 +$$

$$+ \frac{1}{4!}\varphi^4 + \frac{i}{5!}\varphi^5 + \dots =$$

$$= \underbrace{\left(1 - \frac{1}{2}\varphi^2 + \frac{1}{4!}\varphi^4 + \dots\right)}_{\cos\varphi} +$$

$$+ i\underbrace{\left(\varphi - \frac{1}{3!}\varphi^3 + \frac{1}{5!}\varphi^5 + \dots\right)}_{\sin\varphi} =$$

$$= \cos\varphi + i\sin\varphi, \quad \text{q.e.d.}$$

Satz: Es gilt:

$$e^{-i\varphi} = \cos\varphi - i\sin\varphi$$

Beweis:

$$e^{-i\varphi} = e^{i(-\varphi)} =$$

$$= \cos(-\varphi) + i\,(\sin(-\varphi)) =$$

$\cos(-\varphi)$: gerade Funktion, Achsensym. $f(x) = f(-x)$

$\sin(-\varphi)$: ungerade Funktion, Punktsym. $f(-x) = -f(x)$

$= \cos\varphi - i \sin\varphi$, q. e. d.

Def.: (Euler – Form, EF)

Sei $z = |z| (\cos\varphi + i \sin\varphi) \in \mathbb{C}$.

Dann heißt

$z = |z| e^{i\varphi}$

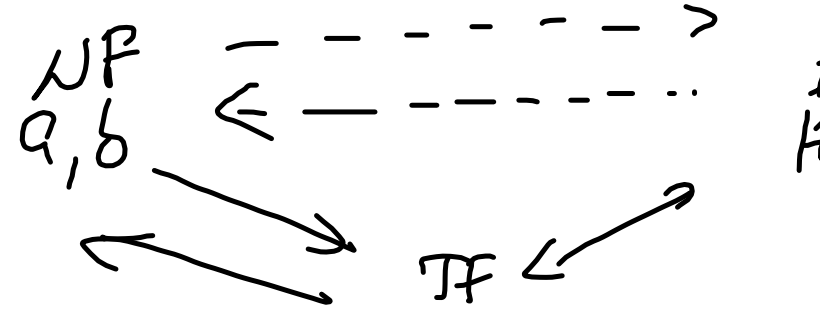
Eulerform (EF) der komplexen Zahl z.

z^* zu z:

Wenn $z = |z| e^{i\varphi}$, dann

$z^* = |z| e^{-i\varphi}$.

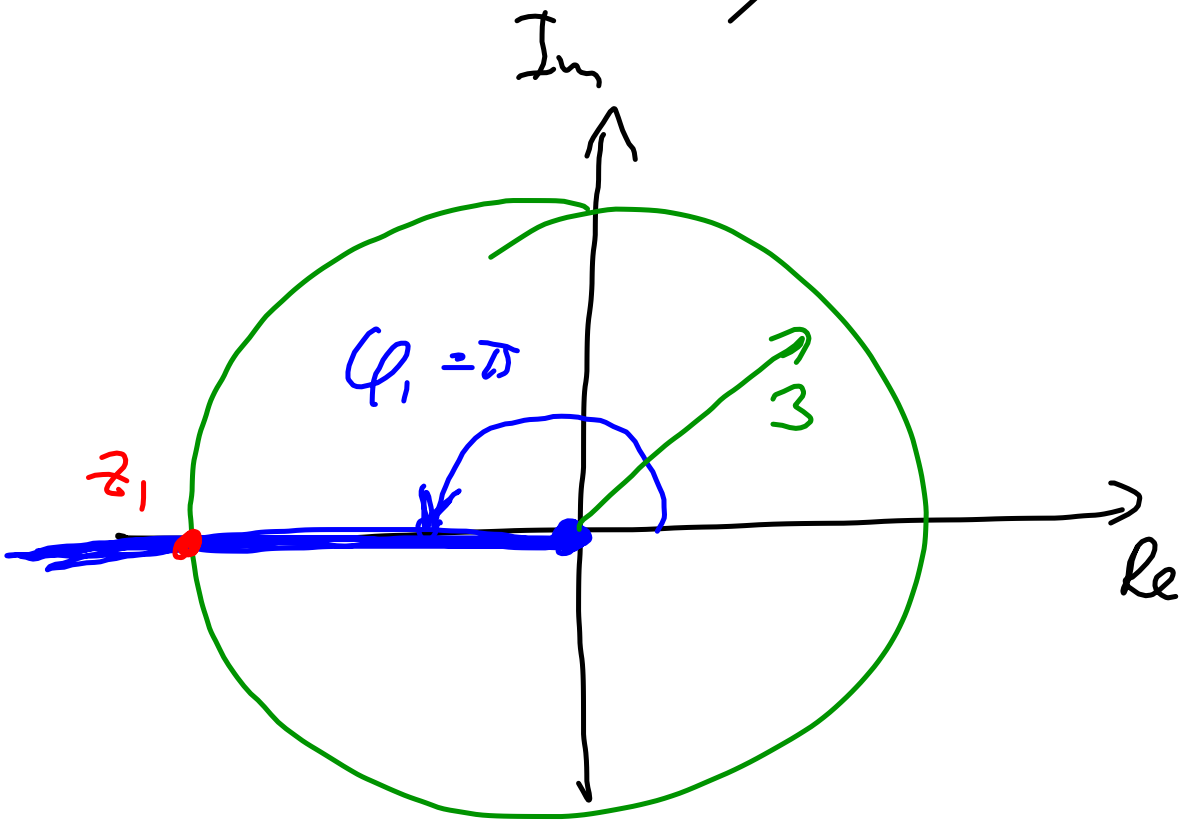
Hinweis: Wir verwende an der Stelle von der TF die EF einer kompl. Zahl. (Dasselbe Koordinatensystem!)

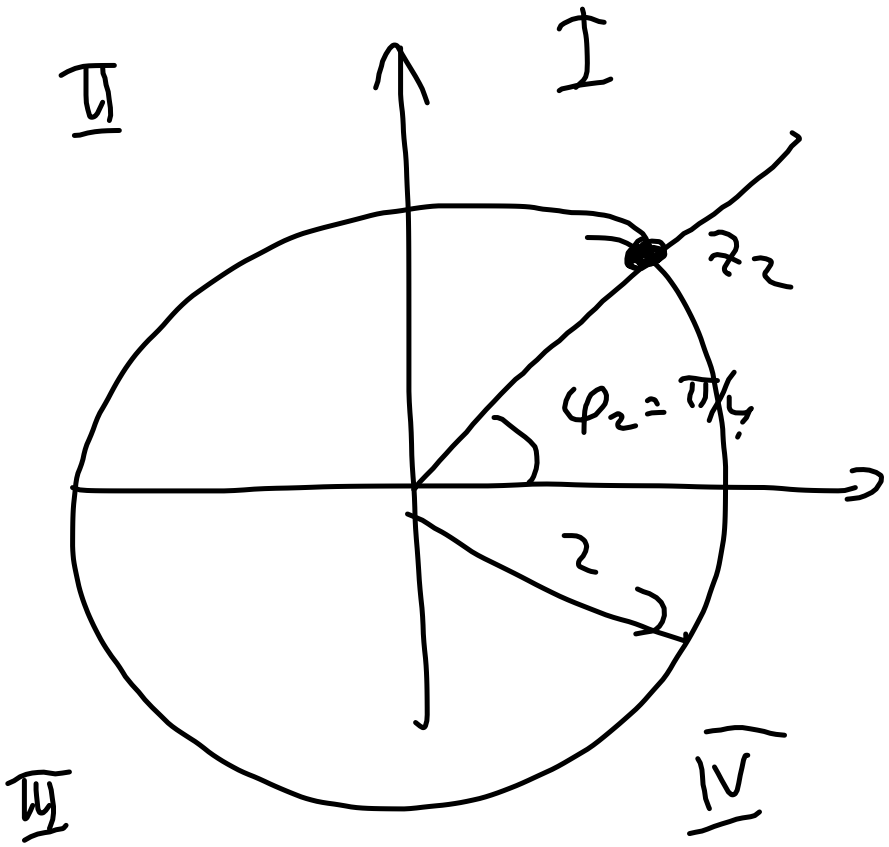


Bsp! Skizziere Sie z!

$z_1 = 3e^{i\pi}$; $|z_1| = 3$, $\varphi_1 = \pi$

$z_2 = 2e^{i\pi/4}$;




II
I
z_2
$\varphi_2 = \pi/4$
z
III
IV

Bsp: $z = 1+2j$ (NF)

$EF(z) - ?$

$\varphi - ?$ (Wie bei TF)

$|z| - ?$

$$|z| = \sqrt{1^2+2^2} = \sqrt{5}$$

$$\varphi = \arctan\left(\frac{2}{1}\right) \approx 63,4° =$$

$$= 0,35 \cdot \pi;$$

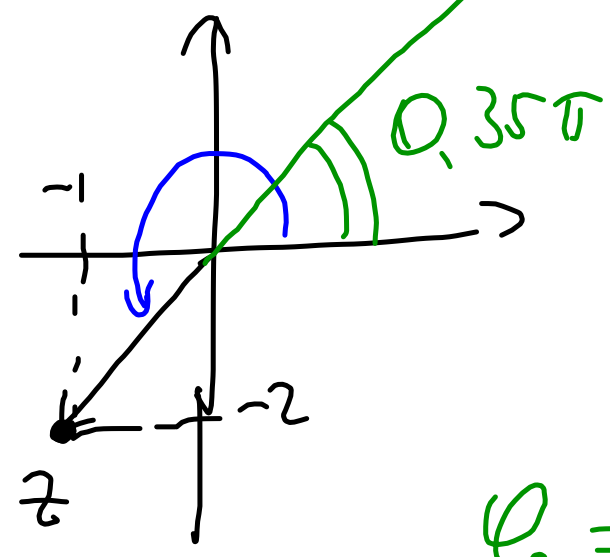
$$z = \sqrt{5}\, e^{i \cdot 0,35 \cdot \pi};$$

$z = -1 - 2i$;

$|z| = \sqrt{1+4} = \sqrt{5}$

$\varphi = \arctan\left(\frac{-2}{-1}\right) =$

$= \arctan 2 = 0{,}35\,\pi$



III

III. Quadrant

$\varphi_z = 0{,}35\pi + \pi$

$z = \sqrt{5}\, e^{i(0{,}35+1)\pi}$

1.3. Rechenoperationen

1.3.1. Ordnungsrelationen

$$z_1 = z_2 \overset{(EF)}{\Longleftrightarrow} (\mathrm{Re}\, z_1 = \mathrm{Re}\, z_2) \wedge (\mathrm{Im}\, z_1 = \mathrm{Im}\, z_2)$$

$$\Updownarrow$$

$$(EF) \quad |z_1| = |z_2| \wedge \varphi_2 = \varphi_1 + 2\pi k, \quad k \in \mathbb{Z}$$

$<\,,>$ gibt's nicht mehr!

1.3.2. Addition und Subtraktion

$+, -$ geht nur, falls z_1, z_2 in NF vorliegen.

Das Ergebnis $(z_1 \pm z_2)$ liegt wieder in NF vor!

$z_1 = a+ib$; $z_2 = c+id$;

$$z_1 \pm z_2 = (a+ib) \pm (c+id) =$$

$$= (a \pm c) + i(b \pm d)$$

Вар: $z_1 = 1+j$ $\quad i=j$

$z_2 = 2e^{j\pi/4}$

$z_1 + z_2 - ?$

z_1 — АЛГ

z_2 — ЕГ $\longrightarrow$ АЛГ

$z_2 = 2e^{j\pi/4} = 2\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

$= 2\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) =$

$= \sqrt{2} + i\sqrt{2}$ (АЛГ)

$$z = z_1 + z_2 = (1+j) + (\sqrt{2} + \sqrt{2}\,i) =$$

$$= (1+\sqrt{2}) + i\,(1+\sqrt{2})$$

$$|z| = \sqrt{2(1+\sqrt{2})^2} =$$

$$= (1+\sqrt{2})\sqrt{2} = \sqrt{2} + 2;$$

I. Quadrant:

$$\varphi = \arctan \frac{1+\sqrt{2}}{1+\sqrt{2}} = \arctan 1 = \frac{\pi}{4}$$

$$z = (\sqrt{2}+2)\, e^{i\,\pi/4}$$

1.3.3. Multiplikation

Sowohl NF als auch EF.

(i) z_1, z_2 in NF $\Rightarrow z_1 \cdot z_2$ in NF

(ii) z_1, z_2 in EF $\Rightarrow z_1 \cdot z_2$ in EF

(i) $z_1 = a + ib$;

$z_2 = c + id$;

$$z_1 \cdot z_2 = (a+ib)(c+id) =$$

$$= ac + ibc + iad + i^2 bd =$$

$$= -1$$

$$= \underbrace{(ac - bd)}_{Re(z_1 \cdot z_2)} + i \underbrace{(bc + ad)}_{Im(z_1 \cdot z_2)}.$$

(ii) $z_1 = |z_1| e^{i\varphi_1}$;

$z_2 = |z_2| e^{i\varphi_2}$;

$z = z_1 \cdot z_2 = \left(|z_1| e^{i\varphi_1}\right)\left(|z_2| e^{i\varphi_2}\right) =$

$= \left(|z_1| \cdot |z_2|\right)\left(e^{i\varphi_1} \cdot e^{i\varphi_2}\right) =$

$= |z_1| \cdot |z_2| \, e^{i\varphi_1 + i\varphi_2} =$

$= |z_1| \cdot |z_2| \, e^{i(\varphi_1 + \varphi_2)}$

Geometrische Bedeutung 24.10.

Bei: Multiplikation komplexer Zahlen Betrag des Ergebnisses ist das Produkt der Beträge der Faktoren, und die Winkel der Faktoren werden zueinander addiert.

Bsp:

$z_1 = 1+i$

$z_2 = 2e^{i\pi/4}$

$z = z_1 \cdot z_2$

(1) NF: $z_1 = 1+i$

$z_2 = 2e^{i\pi/4} = 2\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) =$

$= \sqrt{2} + \sqrt{2}i$

$z = (1+i)(\sqrt{2}+\sqrt{2}i) =$

$$= \sqrt{2} + \sqrt{2}i + i\sqrt{2} + \sqrt{2}\underbrace{i^2}_{=-1} =$$

$$= \sqrt{2} - \sqrt{2} + i\left(\sqrt{2} + \sqrt{2}\right) =$$

$$= 2\sqrt{2}\,i\,;$$

$$|z| = 2\sqrt{2}$$

$$\varphi = \pi/2$$

Bezeichnung: Für z wird oft auch Winkel φ als $\operatorname{Arg} z = \varphi$ bezeichnet

(b) EF: $z_2 = 2e^{i\pi/4}$ $\quad i = j = \sqrt{-1}$

$z_1 = 1 + j$

$|z_1| - ?$ $\quad \varphi_1 - ?$

$$|z_1| = \sqrt{Re^2 z_1 + Im^2 z_1} =$$

$$= \sqrt{2}$$

$$Arg\, z_1 = \varphi_1 - ?$$

I. Quadrant

$$\varphi_1 = \arctan \frac{1}{1} = \arctan 1 = \frac{\pi}{4}$$

$$z_1 = \sqrt{2}\, e^{i\pi/4}$$

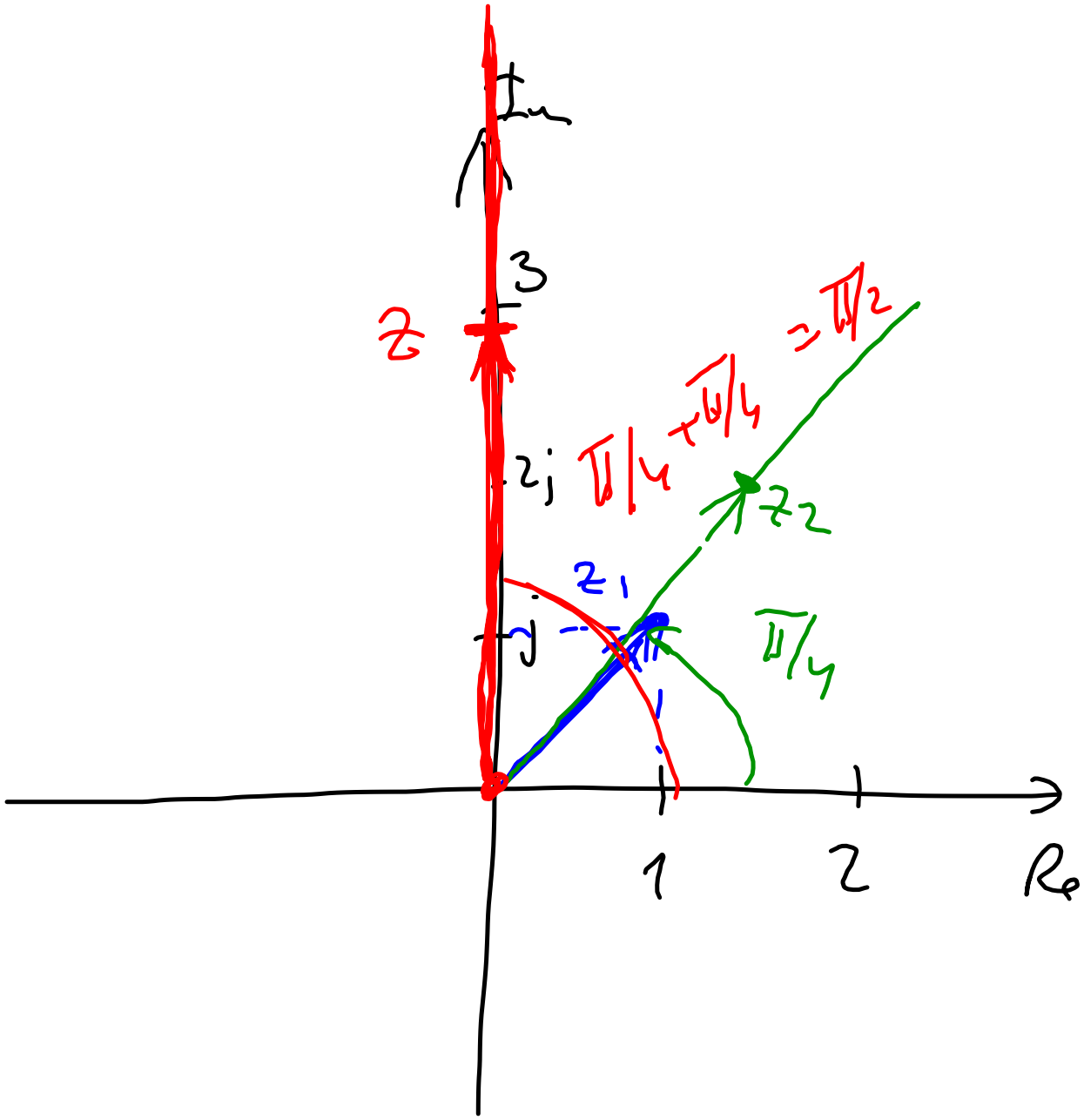
$$z = z_1 \cdot z_2 = \left(\sqrt{2}\, e^{i\frac{\pi}{4}}\right)\left(2\, e^{i\pi/4}\right) =$$

$$= 2\sqrt{2}\, e^{i\left(\frac{\pi}{4}+\frac{\pi}{4}\right)} =$$

$$= 2\sqrt{2}\, e^{i\pi/2}$$

$$|z| = 2\sqrt{2}$$

$$\varphi = \operatorname{Arg}\varphi = \pi/2$$


Im
3
z
2j
π/4
+π/4
=π/2
z2
z1
-j
π/4
1
1
2
Re

(3) $z_2 \cdot z_1^*$

$|z_1| = |z_1^*|$

$\operatorname{Arg} z_1 = -\operatorname{Arg} z_1^*$

$\operatorname{Re} z_1 = \operatorname{Re} z_1^*$

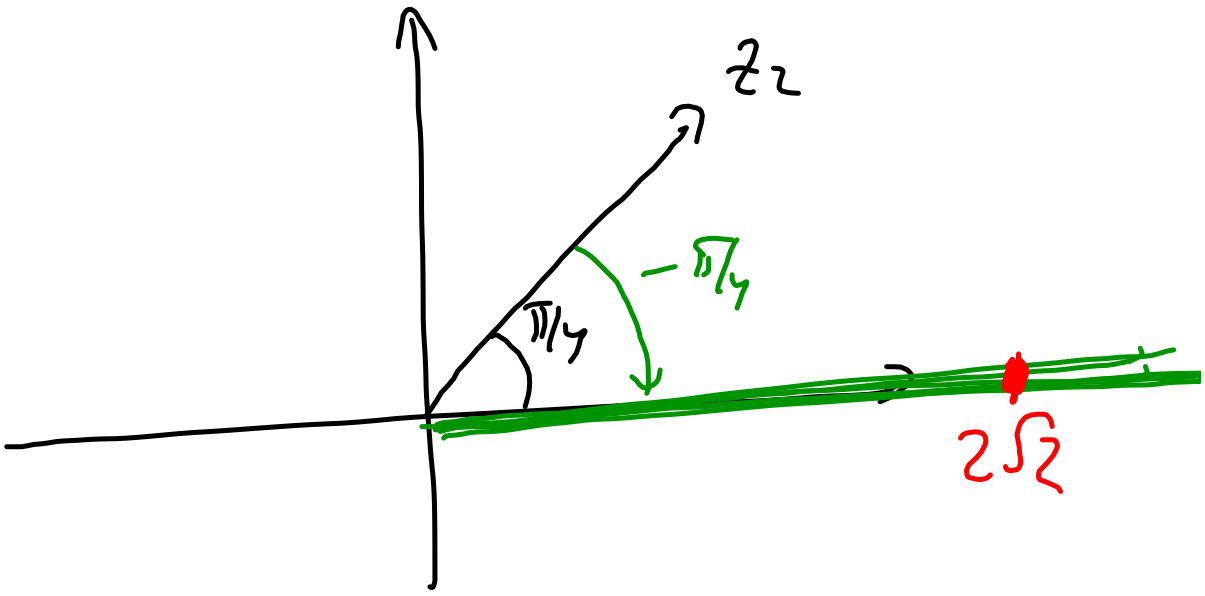
$\operatorname{Im} z_1 = -\operatorname{Im} z_1^*$

EF: $z_1 = \sqrt{2}\, e^{i\pi/4}$

$z_1^* = \sqrt{2}\, e^{-i\pi/4}$

$$z_2 \cdot z_1^* = 2 e^{i\pi/4} \cdot \sqrt{2}\, e^{-i\pi/4} =$$

$$= 2\sqrt{2}\, e^{i(\pi/4 - \pi/4)} = 2\sqrt{2}$$



1.3.4. Division

$$\frac{NF}{NF} = NF$$

$$\frac{EF}{EF} = EF$$

1.3.4 1. Dividna in Nf-

Bsp:

$$z = \frac{1+2j}{1-3i} \cdot \underbrace{\left(\frac{1+3j}{1+3j}\right)}_{=1} =$$

$$= \frac{(1+2j)(1+3i)}{(1-3i)(1+3i)} =$$

$$= \frac{1-6+5i}{10} = \frac{-5+5i}{10} =$$

$$= -\frac{1}{2} + \frac{1}{2}i$$

$z \cdot z^*$

Für $z = a + ib$;

$$z \cdot z^* = (a+ib)(a-ib) =$$

$$= a(a-ib) + ib(a-ib) =$$

$$= a^2 - iab + iab - i^2 b^2 =$$

$i^2 = -1$

$$= a^2 + b^2 = |z|^2 = |z^*|^2$$

$$\boxed{z \cdot z^* = |z|^2 = |z^*|^2}$$

Allgemeine Schritte:

$$z = \frac{z_1}{z_2} =$$

(1) Erweitern mit der komplex konjugierten Zahl zum Nenner: z_2^*

$$= \frac{z_1 \cdot z_2^*}{z_2 \cdot z_2^*} =$$

(2) Multipliziere $z_1 \cdot z_2^* =$ in NF $= a + ib$

(3) $z_2 \cdot z_2^* = |z_2|^2 \in \mathbb{R}$

$$= \frac{a+ib}{|z_2|^2} =$$

$$= \frac{a}{|z_2|^2} + i\frac{b}{|z_2|^2}$$ ist NF

1.3.4.2. Division in EF

Bsp:

$$z = \frac{1+2j}{1-3j} \qquad \frac{EF}{EF}$$

$z_1 = 1+2j \longrightarrow EF$

$$|z_1| = \sqrt{Re^2 z_1 + Im^2 z_1} =$$

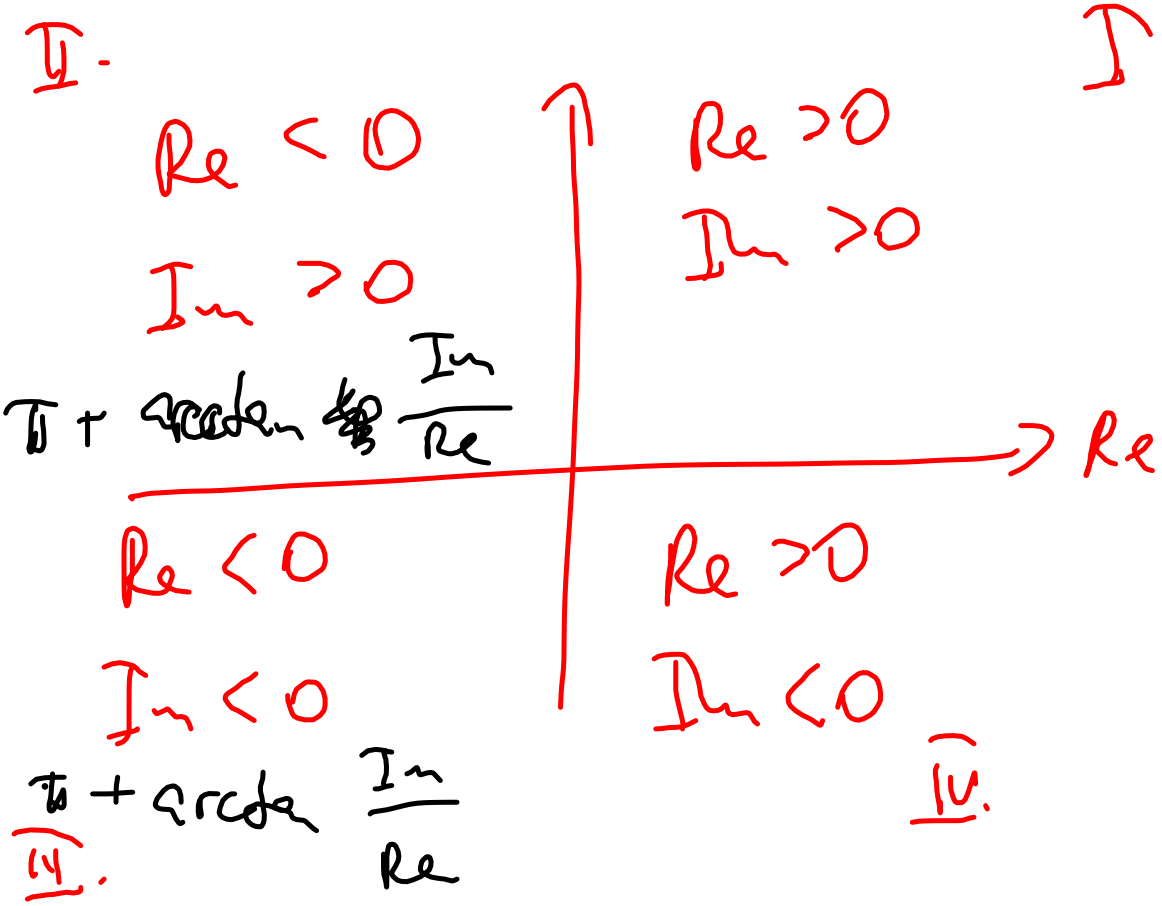
$$= \sqrt{1^2+2^2} = \sqrt{5}$$

$$Arg\, z_1 = \varphi_1 = \arctan\frac{2}{1} = 1{,}107$$

$|z_2| = \sqrt{10}$ Weil in IV. Qu.

$\varphi = \arg z_2 = \arctan(-3) =$

$= -1{,}249$



$$\frac{z_1}{z_2} = \frac{|z_1|\, e^{i\varphi_1}}{|z_2|\, e^{i\varphi_2}} =$$

$$= \frac{|z_1|}{|z_2|}\, e^{i\varphi_1 - i\varphi_2} =$$

$$= \frac{|z_1|}{|z_2|}\, e^{i(\varphi_1 - \varphi_2)}$$

Allgemeine Formel:

$$\frac{z_1}{z_2} = \frac{|z_1|}{|z_2|}\, e^{i(\varphi_1 - \varphi_2)}$$

$$\frac{z_1}{z_2} = \frac{\sqrt{5}}{\sqrt{10}} e^{i(\varphi_1 - \varphi_2)} =$$

$$= \frac{1}{\sqrt{2}} e^{i(1.107 - (-1.249))} =$$

$$= \frac{1}{\sqrt{2}} e^{i \cdot 2.356} \quad ; \quad (\text{EF})$$

NF: $\operatorname{Re} z = \frac{1}{\sqrt{2}} \cos(2.356)$

$\operatorname{Im} z = \frac{1}{\sqrt{2}} \sin(2.356)$

$\operatorname{Re} z \approx -\frac{1}{2}$

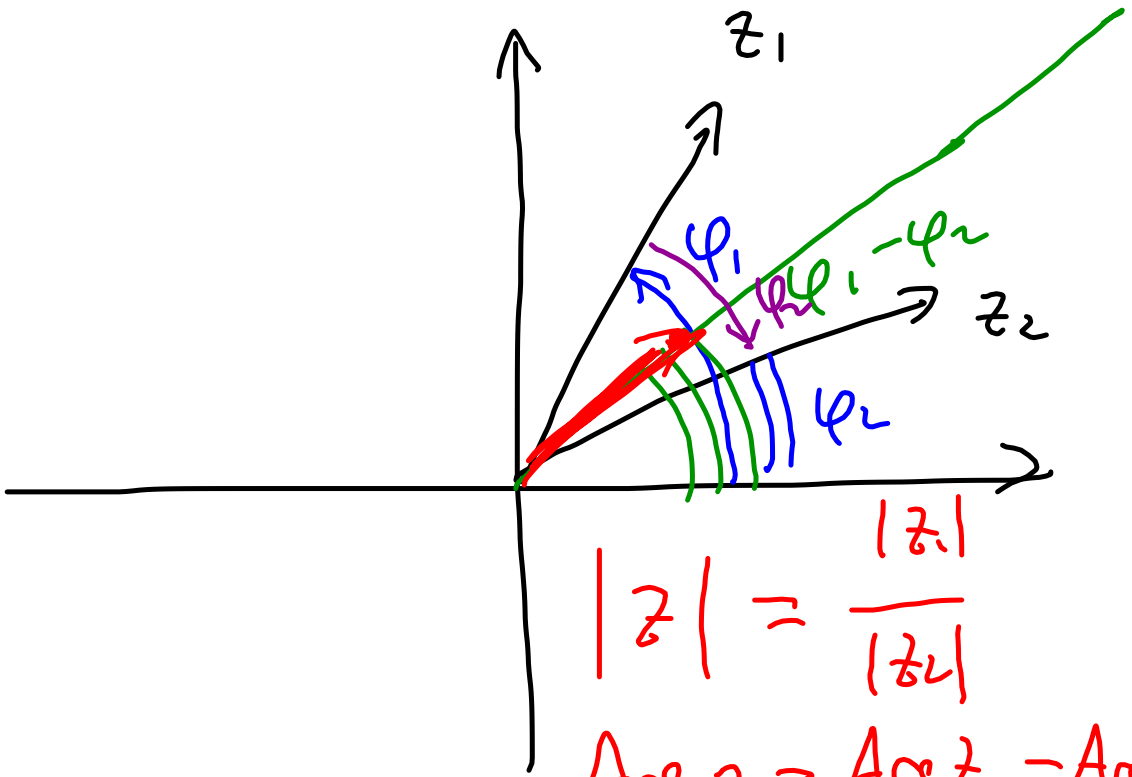
$\operatorname{Im} z \approx \frac{1}{2}$

$$\boxed{z = -\frac{1}{2} + \frac{1}{2} i}$$

Dieselbe Zahl wie im Bsp. NF !

Geometrische Bedeutung:

$$z = \frac{z_1}{z_2} = \frac{|z_1|}{|z_2|}\, e^{i(\varphi_1 - \varphi_2)}$$



$$|z| = \frac{|z_1|}{|z_2|}$$

$$\operatorname{Arg} z = \operatorname{Arg} z_1 - \operatorname{Arg} z_2$$

$$= \varphi_1 - \varphi_2$$

Wurzel einer komplexen Zahl

Formel von Moivre

Herr in EF (oder TF)

$z \in \mathbb{C}$

$\sqrt[n]{z}$:

Satz: $\forall z \in \mathbb{C} \; \forall n \in \mathbb{N}$

gibt es genau n verschiedene

komplexe Wurzeln $\sqrt[n]{z}$,
die sich periodisch
wiederholen:

$$\sqrt[n]{z} = \sqrt[n]{|z|}\, e^{i\left(\frac{\varphi}{n} + 2\pi\frac{k}{n}\right)},$$

$$k = 0, 1, 2, \ldots, n-1$$

und $\varphi = \operatorname{Arg} z$.

Diese Wurzeln sind
gleichmäßig auf Kreis
mit Radius $\sqrt[n]{|z|}$ verteilt.

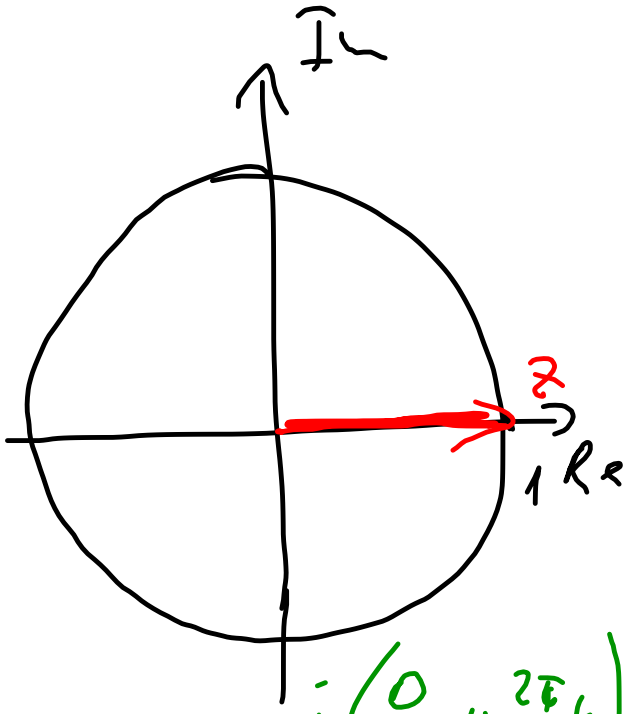
Bsp!

$\sqrt[3]{+1}$ — ?

$z = +1$

$|z| = 1$

$\operatorname{Arg} z = 0$



$i\left(\frac{0}{3} + \frac{2\pi}{3}k\right)$!

$\sqrt[3]{z} = \sqrt[3]{1} = \sqrt[3]{|1|}\, e$

$k = 0, 1, 2$

$(k=0)$ $z_1 = 1e^{i(0+0)} = 1$; $i\,2\pi/3$

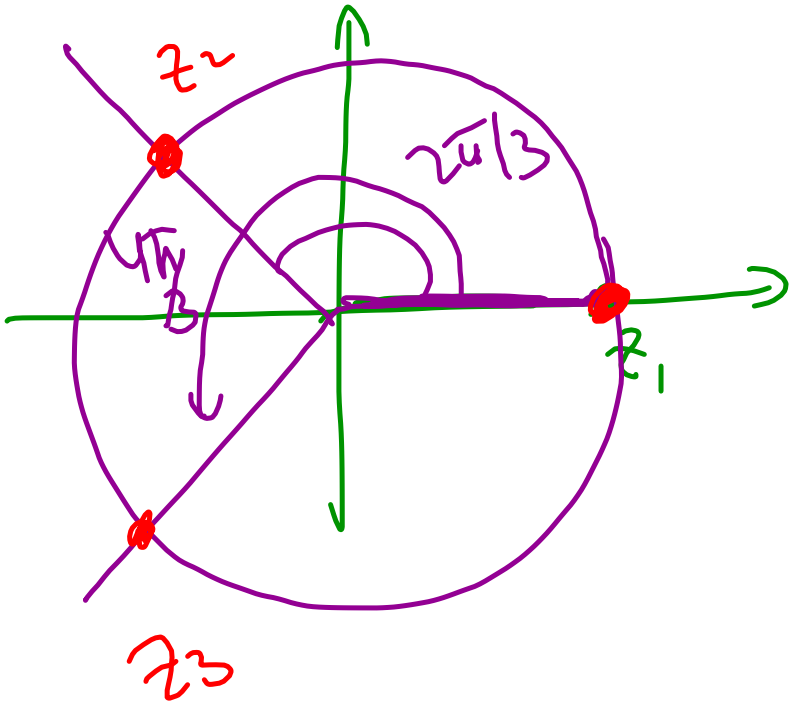
$(k=1)$ $z_2 = 1 \cdot e^{i\left(0+\frac{2\pi}{3}\right)} = e$; $i\,4\pi/3$

$(k=2)$ $z_3 = 1 \cdot e^{i\left(0+\frac{4\pi}{3}\right)} = e$

$$1 \cdot e^{i \cdot 0} = 1 \cdot e^{i \, 0 \cdot 2\pi/3} =$$

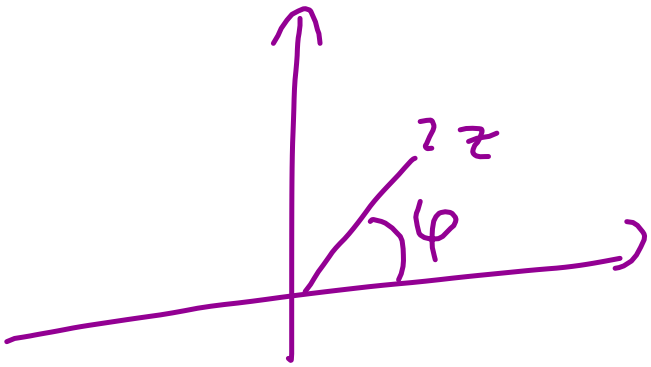
$$= 1 \cdot e^{i \cdot \frac{6\pi}{3}}$$

$$\frac{6\pi}{3} = 2\pi$$



arg z 3d 2π -

Periodisch



Arg z = φ

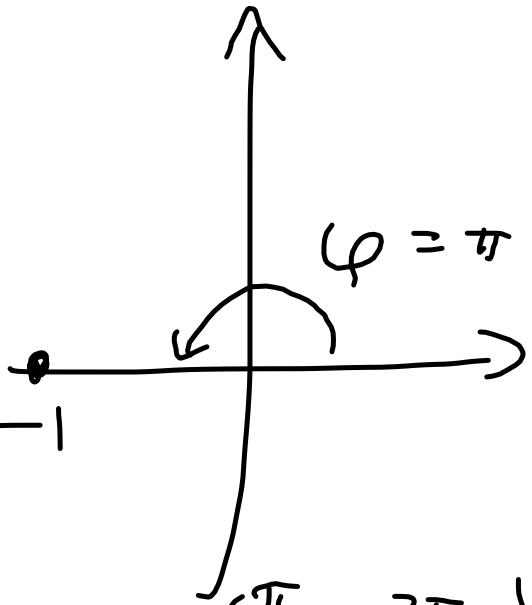
arg z = φ + 2πk, k ∈ ℤ

(2) $\sqrt[4]{-1} \notin \mathbb{R}$

$z = -1$

$|z| = 1$

$\operatorname{Arg} z = \varphi = \pi$



$n = 4$

$\sqrt[4]{-1} = \sqrt[4]{|-1|}\, e^{i\left(\frac{\pi}{4} + \frac{2\pi}{4}k\right)}$,

$k = 0, 1, 2, 3$

$$z_{k=0} = \sqrt[4]{1}\, e^{i\left(\frac{\pi}{4} + \frac{2\pi}{4}\cdot 0\right)} =$$

$$= 1 \cdot e^{i\,\pi/4} = e^{i\,\pi/4}$$

$$z_{k=1} = e^{i\left(\frac{\pi}{4} + \frac{2\pi}{4}\cdot 1\right)} =$$

$$= e^{i\left(\frac{\pi}{4} + \frac{\pi}{2}\right)} = e^{i\,\frac{3\pi}{4}}$$

$$z_{k=2} = e^{i\left(\frac{\pi}{4} + \frac{2\pi}{2}\right)} = e^{i\,\frac{5\pi}{4}}$$

$$z_{k=3} = e^{i\left(\frac{\pi}{4} + \frac{3\pi}{2}\right)} = e^{i\,\frac{7\pi}{4}}$$

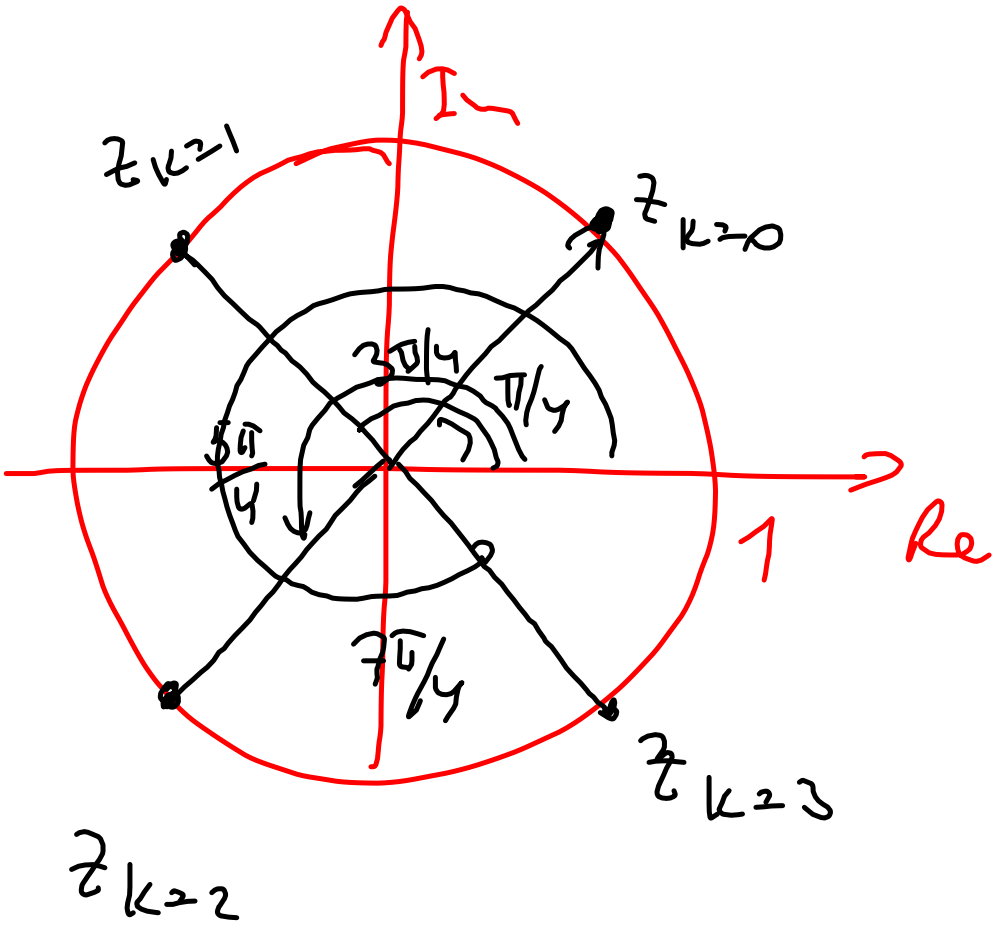
$K=4$?

$$z_{K=4} = e^{i\left(\frac{\pi}{4} + 4\cdot\frac{\pi}{2}\right)} =$$

$$= e^{i\left(\frac{\pi}{4} + 2\pi\right)} =$$

volle Periode

$$= e^{i\pi/4} = z_{K=0}$$


Im
Re
1
$z_{k=1}$
$z_{k=0}$
$z_{k=2}$
$z_{k=3}$
3π/4
π/4
5π/4
7π/4